Modeling Voter Responses to Campaign Expenditures
Abstract:

This paper investigates the effect of campaign expenditures by two candidates competing for a simple majority in a popular election. It begins with an examination of past empirical and theoretical papers. These papers posit a variety of assumptions about an aggregate function that maps the candidates’ expenditures to their probability of winning the election. This paper, in contrast, derives the properties of the aggregate probability of winning function from primitive assumptions about the influence of campaign spending on the behavior of individual voters. In the model developed here, a candidate must consider not only the spending level of his opponent, but also the initial distribution of voters’ political preferences, the sensitivity of those preferences to spending, and the relationship between a voter’s strength of political preference and his probability of casting a vote. Marsden (2008) has shown that the existence of equilibrium winning strategies for each candidate, in an Electoral College election system, depends carefully on the properties of the winning function for each state. The goal of this analysis is to identify sets of assumptions about the individual-level model that generate an aggregate probability of winning function consistent with either the Snyder (1989) or Marsden (2008) functions.
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1. Introduction

Money is the resource that fuels candidates’ campaign efforts. The cost of running a U.S. presidential campaign can total hundreds of millions of dollars. The majority of this money goes toward various forms of advertising, which are meant to reach the voters and sway their political preferences in a candidate’s favor. Understanding how individual voters are affected by the spending of these campaign resources is crucial. This information can be used to find the probability of winning a state-level popular vote given certain expenditures. Candidates would certainly be interested to know the level of returns expected for each dollar they spend. Most papers related to two-candidate voting games use very simplified functional forms to describe how an electorate will behave as a whole in response to campaign spending of a candidate. The broad characterization of many voters is often defined arbitrarily, ignoring how exactly the individual voters collectively might actually bring about the observance of results in line with these functional forms. It is important not to neglect the independence of each voter in his or her choice of how to vote.

Thinking about the individual voters themselves, as opposed to large groups of voters, allows incorporation of each voter’s personal behavior and political preferences. Party bias can be included much more specifically, as it can be attached to each individual. Analyzing effects of spending from this micro-perspective can show a great deal more about the most basic influences of campaign spending, which occur in each single vote cast. Most metrics that would describe the policy preferences and behavior of a large group cannot be distinguishing enough to properly characterize the population’s
true political disposition. For example, consider a state that contains only people with very polar political preferences. If they are divided almost evenly between the two parties, an election race could look very close. The state could be labeled as a politically moderate battleground state deserving of increased campaign spending. However, upon closer inspection it could become apparent that the cost of increasing vote share by trying to convert staunch democrats or republicans is too costly. Looking at individual voters, it becomes clearer that, based on the expected behavior of extreme liberals or conservatives, increased campaign spending efforts to change their political preferences might be relatively ineffective. There are certainly effects at the individual voter level in response to campaign spending that would be unnoticed by larger scale analysis.

This paper provides more information on this issue by presenting a model of how campaign spending affects individual voters’ political preferences and voting behavior. From this micro model, the implications of spending with respect to winning a popular election can be examined. The focus of the analysis is on how the properties of the aggregate probability of winning function depend on the assumptions made in the individual-level model. Each assumption might represent a particular situation or bias. Every collection of these specifications can have a unique story to tell with respect to the height and the shape of the objective function.

2. Review of Past Literature

Campaign spending can be used in a number of ways, such as for buying up advertising space to increase a candidate’s popularity or even to tarnish an opponent’s
image. A number of studies have shown that campaign spending does increase vote share for a candidate. Duquette, Caudill, and Mixon Jr. (2008), for example, look at empirical data from U.S. House open seat elections from 1990 to 2004. They find that election outcomes are highly responsive to candidates’ spending ratios. For example, the higher is the ratio of Democrat spending to Republican spending, the greater is the Democratic candidate’s chance of winning. It also suggests that, even for candidates already outspending their opponents, additional campaign spending has nontrivial returns for probability of winning the election. The positive relationship between campaign spending and probability of winning an election is a critical element of the model in this paper.

Stratmann (2009) examines the apparent ineffectiveness of campaign spending for incumbents in congressional elections as compared to the significant return on campaign spending for the opponent. He blames omitted variable bias in previous work for producing this counterintuitive finding. His analysis shows that this result comes from previous studies having neglected to take into account varying costs of advertising across districts and the differences in elasticity of demand for campaign advertising between incumbents and challengers. Stratmann finds that incumbents have a significantly more inelastic demand for advertising, which can lead to a tendency of incumbents to spend more in districts with much higher costs of advertising. With the increased costs, the incumbents are receiving a lower quantity of advertising for their money, which helps to explain why they seem to be receiving less return per dollar spent compared to their opponents. Accounting for these differences in advertising price reveals that campaign spending for incumbents actually does have a positive marginal product. The model in
this paper does not make a distinction between incumbents and non-incumbents, so it is useful to have evidence their spending returns are not as different as they would appear.

Rekkas (2007) has used Canadian federal election data to find very interesting information about spending and voter behavior. She observed that increased spending of one party would increase the number of votes received by its candidate, but sometimes at the expense of another party. This implies that there are two major effects of spending in how it influences the electorate. The first is that increased spending by one candidate will make people more likely to cast a vote in his favor rather than abstain from voting. Secondly, it appeared that campaign spending could redistribute voters across parties. This means that some voters will decide not to vote for the candidate they originally preferred, and, in response to the effects of increased spending, will instead vote for the other candidate. The result is essentially stealing votes away from the other candidate, as opposed to just solidifying the voters who were already in favor. The model in this paper incorporates the possibility of observing such effects.

A closely related paper by Herrera, Levine, and Martinelli (2007) does take a closer look at individual voters. In their model, voters’ preferences are indexed on the interval $[0,1]$, with lower numbers favoring the democratic candidate and higher numbers favoring the republican candidate. Campaign spending is thought of in terms of campaign effort, which takes on a value from the interval $[0,1]$. The model also allows for candidates to choose specific policy platforms. Adjusting platforms can change the number of voters who favor a candidate, as voters will tend to side with a candidate running a platform similar to their policy preference. Also included in their model is
accuracy of campaign targeting, which can affect voter turnout when combined with level of campaign effort.

This paper describes a direct effect of campaign spending on voters’ policy preferences and probability of voting, which splits with the Herrera, Levine, and Martinelli (2007) model. In their model, spending or “campaign effort” does not have any influence on which candidate a voter might favor. The only thing that does affect which candidate a voter prefers is the choice of political platforms (It should be noted that the model in this paper is not concerned with political platforms chosen by the candidates). Based on findings from Rekkas (2007) and other studies looking at the influence of campaign spending on voters, the voters’ preferences for one candidate or the other should be sensitive to spending. The model in this paper also allows for analysis involving varying distributions of voters’ preferences. This is opposed to making assumptions that preferences of the electorate are uniformly distributed over some interval, such as in Herrera, Levine, and Martinelli (2007).

A paper by Colontoni, Levesque, and Ordeshook (1975) looks at how candidates optimally distribute their resources across states in an Electoral College election. Although the paper does not discuss behavior of individual voters, it does recognize the importance of how one defines the probability of winning individual states, which is really dependent on voters. The authors introduce the idea that there could be a multitude of unique functional forms used to define the probability of winning a state, which can each affect candidates’ resource distributions across states. These functional forms are designed to capture certain characteristics of the electorate’s response to spending. For example, there might be marginal returns to spending that are always decreasing. There
might also be, for a different state, marginal returns to spending that are increasing to a point, and then decreasing. A number of papers have chosen to accept specific functional forms for use in analysis. This paper explores how using various parameters to define behavior of individual voters can actually produce these functional forms.

Other macro-oriented research papers by Snyder (1989) and Marsden (2008) inspired the focus of this paper on voters. Such papers skipped over using a micro-level model involving individual voters in favor of a model that ties probability of winning an election directly to spending using a particular functional form. Snyder used the framework of congressional elections, where his functional form gave the probability of winning each single seat in Congress. The goal for each political party was to win the majority of the seats. Figure 1 pictures an example of the Snyder functional form. The probability of a party “A” winning a specific seat in Congress in on the vertical axis, with the party’s spending level “a” on the horizontal axis for a fixed opponent spending level. This functional form featured returns to spending that were positive but always decreasing within each specific Congress seat election. Snyder showed that, under these conditions, equilibrium levels of spending by each party exist. Marsden used the Electoral College system for his analysis. Presidential candidates competed to win various states so that they would be awarded those states’ electoral votes. The candidate who wins the majority of electoral votes wins the presidential election. Marsden’s functional form gave the probability of winning each state. Figure 2 shows an example of this functional form. The probability of candidate “A” winning a state is on the vertical axis, with his spending level “a” on the horizontal axis for a fixed opponent spending level. It used the idea that returns to a candidate’s campaign spending were largest when
his level of spending was closest to that of his opponent. So, the fact that the graph is steepest at $a=5$ implies that the opponent is spending 5. Using this functional form, Marsden showed that no pure strategy equilibrium exists in the Electoral College campaign spending model.

In the Snyder and Marsden examples, existence of equilibria depends crucially on the specific functional forms for defining the probability of winning states, or seats in Congress. These functional forms, however, were defined arbitrarily. This paper develops a micro-level model of individual voters to investigate how these functional forms might be explained by voter behavior in response to spending. Snyder and Marsden both used functional forms as simplified representations of popular vote elections that contribute to a larger scheme, which would be getting the majority of seats in Congress or winning the majority of electoral votes. The model in this paper can be used to establish a specific way of describing the probability of winning a popular vote based on individual voters. This means that the probability of winning can be directly compared to the probability of winning a seat in Congress, which is defined by Snyder’s function form, or the probability of winning a state, which is defined by Marsden’s functional form. The results can give some evidence in favor of one of these functional forms or perhaps suggest a new form that behaves differently.

3. Model

Two candidates, A and B, are competing to win a simple majority in a popular election. An electorate contains $n$ voters, indexed $i = 1, \ldots, n$, who each have the potential
to cast a single vote. The political preference of voter $i$ is given by a real number $v_i$. A voter with $v_i$ equal to zero is considered indifferent and does not prefer either candidate, a voter with $v_i$ less than zero prefers Candidate A, and a voter with $v_i$ greater than zero prefers Candidate B. The smaller is $v_i$, the more preferred is Candidate A. The larger is $v_i$, the more preferred is Candidate B.

3.1 The Probability of Voting

The magnitude of a voter’s political preference, $v_i$, affects the probability that he casts a vote for his preferred candidate. In particular, the stronger is a voter’s political preference (in other words, the larger is the absolute value of $v_i$), the more likely is the voter to cast his vote for the candidate he prefers. Voters always have a zero probability of voting against their preferred candidates. Each voter either votes for his preferred candidate or abstains. A voter with $v_i = 0$ always abstains. Political preference and a voter’s probability of voting can be related in a few different ways, the first of which is defined:

$$\alpha_i = \frac{v_i}{(v_i - c)} \quad \text{if } 0 \geq v_i \quad (1a)$$

$$\alpha_i = 0 \quad \text{if } 0 < v_i$$

$$\beta_i = \frac{v_i}{(v_i + c)} \quad \text{if } 0 \leq v_i \quad (1b)$$

$$\beta_i = 0 \quad \text{if } 0 > v_i$$
A graph of (1a) and (1b) can be seen in *Figure 3* and *Figure 4* respectively. \( \alpha_i \) is the probability that voter \( i \) votes for Candidate A, \( \beta_i \) is the probability that voter \( i \) votes for Candidate B, and \( c \) is any positive real constant. In this example, both candidates are facing decreasing returns to strength of political preference in their favor, with the returns being the increase in probability of receiving a vote from voter \( i \). The larger is \( c \) the less are the returns to strength of political preference for all \( v_i \).

Here is another option:

\[
\begin{align*}
\alpha_i &= \begin{cases} 
0 & \text{if } 0 \leq v_i \\
\frac{v_i}{-c} & \text{if } -c \leq v_i < 0 \\
1 & \text{if } -c > v_i 
\end{cases} \quad (2a) \\
\beta_i &= \begin{cases} 
0 & \text{if } 0 \geq v_i \\
\frac{v_i}{c} & \text{if } c \geq v_i > 0 \\
1 & \text{if } c < v_i
\end{cases} \quad (2b)
\end{align*}
\]

A graph of (2a) and (2b) can be seen in *Figure 5* and *Figure 6* respectively. This time both candidates are facing constant returns to strength of political preference in their favor until the probability of voting hits one. For the \( c \) in these equations, again larger values imply lesser returns (a more horizontal slope when graphing probability of voting on strength of political preference). It follows that \( c \) will also determine the \( v_i \) at which the probability of voting hits one, with higher values of \( c \) implying a higher \( v_i \) needed to reach a probability of voting equal to one. At this point, for any stronger political preference the voter will vote with probability of one.

A third option is:
\[ \alpha_i = 0 \quad \text{if} \quad 0 \leq v_i \]  
\[ \alpha_i = \frac{2k}{v_i} \quad \text{if} \quad -\frac{2k}{\sqrt{c}} \leq v_i < 0 \]  
\[ \alpha_i = 1 \quad \text{if} \quad -\frac{2k}{\sqrt{c}} > v_i \]

\[ \beta_i = 0 \quad \text{if} \quad 0 \geq v_i \]  
\[ \beta_i = \frac{2k}{v_i} \quad \text{if} \quad \frac{2k}{\sqrt{c}} \geq v_i > 0 \]  
\[ \beta_i = 1 \quad \text{if} \quad \frac{2k}{\sqrt{c}} < v_i \]

A graph of (3a) and (3b) can be seen in Figure 7 and Figure 8 respectively. The value of \( k \) is any positive integer. It determines the rate at which the value of the function will change exponentially until probability of voting reaches one, with a larger \( k \) implying a greater rate. Again, the larger is \( c \), the more horizontal the slope of the function and the lesser the return. This time both candidates are facing increasing returns to strength of political preference in their favor until the probability of voting hits one. At this point, for any stronger political preference the voter will vote with probability of one.

### 3.2 The Shift in Voters’ Preferences

In this model, campaign spending has the effect of shifting each individual voter’s political preference \((v_i)\). If Candidate A increases spending, it will shift every voter’s \( v_i \) to the left towards smaller values. This can have a few different effects. First, if that voter’s \( v_i \) begins negative and shifts further negative, it increases the probability of that voter voting for Candidate A. Second, if the voter’s \( v_i \) begins positive and shifts to a lower
positive number, it decreases the probability that the voter will vote for Candidate B. Third, if the voter’s $v_i$ begins positive and shifts to a negative number, the probability of the voter voting for Candidate B decreases to 0 and the probability of the voter voting for Candidate A increases from 0 to a positive value. Expenditures by Candidate B have symmetrical effects. Increased spending by Candidate B will shift every voter’s $v_i$ to the right towards larger values. This can increase the probability that a voter votes for him, decrease the probability that a voter votes for his opponent, or do both in having a voter switch preferred candidate.

Initially it is assumed that moderate voters (those with $v_i$ close to 0) should be influenced the most by campaign spending. Intuitively, a voter without strong preferences toward one party could be manipulated most easily. Staunch supporters of Candidate A with very small values of $v_i$ and staunch supporters of Candidate B with very large values of $v_i$ should be influenced the least by campaign spending. So, a voter with a $v_i$ close to 0 will shift further in response to candidate spending than would a voter with a $v_i$ far from 0. Also, campaign spending should have decreasing returns to scale. Only so many advertisements can be run before they lose their effectiveness or the target audience has already been reached. The effect of campaign spending is modeled this way:

$$\Delta v_i = \frac{s(b - a)}{(z + a)(z + b)(1 + v_i^2)}$$  \hspace{1cm} (4)

In this equation describing what I call the “shift function”, $\Delta v_i$ represents the change in political preference for voter $i$. The level of spending by Candidate A is $a$, the level of spending by Candidate B is $b$, and $s$ and $z$ are positive real constants. Increasing $s$ will magnify the shift for each voter, while increasing $z$ will decrease the shift for each voter.
It should be noted that the largest “shifts” in voter political preferences occur when $v_i$ is close to 0. Also, for any $v_i$ the “shift” will always be positive if Candidate B outspends Candidate A and negative if Candidate A outspends Candidate B. In other words, the shift in political preferences of the voters will always be in favor of whichever candidate spends the most. Figure 9 shows the shift for each $v_i$ using (4), with Candidate B outspending Candidate A by some amount causing all shifts to be positive. For a given opponent spending level, the more a candidate outspends his opponent, the greater is the shift. There is no shift if the spending levels of Candidate A and Candidate B are equal. The marginal returns to spending are decreasing for each candidate, with the returns being the shift in voter political preferences in their favor.

One might also want to consider a function such that the distance a voter’s preference shifts depends on the spending levels of the two candidates, but not on the voter’s initial preference ($v_i$). An example of such a function is:

$$\Delta v_i = \frac{sv_i(b-a)}{a+b+z} \quad (5)$$

Finally, though it would be much less plausible, we can also consider a function that shifts voters with the strongest preferences the most:

$$\Delta v_i = \frac{s|v_i|(b-a)}{a+b+z} \quad (6)$$

Graphs of equations (5) and (6), each with Candidate B outspending Candidate A, can be seen in Figure 10 and Figure 11 respectively. In both equations, increasing $s$ magnifies the shift while increasing $z$ decreases it. There are decreasing marginal returns to spending.
3.3 The Probability of Winning

In order to guarantee the win in the election, a candidate must receive more votes than his opponent. If both candidates get the same number of votes, a winner is chosen at random. Given which candidate each voter prefers and the probability that each voter will vote for his preferred candidate, a numerical probability of winning can be calculated.

There are \( \sum_{i=0}^{n} 1 + i \) feasible combinations of different vote totals for the two candidates in an election simulation. For some combinations there would be a tie and for others one candidate wins outright. Each combination has a specific probability of occurring and can be achieved in a number of ways using votes from various collections of voters. If an electorate contains \( n \) voters, there are \( \binom{n}{m} \) different collections of voters that could give a candidate exactly \( m \) votes. Summing the probabilities of each collection occurring gives the probability of a candidate receiving exactly \( m \) votes, which will be very useful for calculating the probability of winning the election.

Consider the following equation:

\[
\rho^A_m = \sum_{i_1=1}^{n-m+1} \alpha_{i_1} \times \sum_{i_2>i_1} \alpha_{i_2} \times \sum_{i_3>i_2} \alpha_{i_3} \times \ldots \times \sum_{i_m>i_{m-1}} \alpha_{i_m} \times \prod_{j \neq i_1, \ldots, i_m} (1 - \alpha_j) \quad (7)
\]

Since \( \alpha_i \) is the probability that voter \( i \) votes for Candidate A, \((1 - \alpha_i)\) is the probability that voter \( i \) does not cast a vote. \( \rho^A_m \) is the probability that Candidate A receives exactly \( m \) votes, so \( \sum_{i=0}^{n} \rho^A_i = \sum_{i=0}^{n} \rho^B_i = 1 \) in an electorate with \( n \) voters. The probability that
Candidate A receives more votes than his opponent would be 
\[
\sum_{i=1}^{n} \rho_{i}^{A} \ast \left( \sum_{j=0}^{i-1} \rho_{j}^{B} \right) \]
and the probability of a tie in number of votes received for each candidate would be 
\[
\sum_{i=0}^{\lfloor n/2 \rfloor} \rho_{i}^{A} \ast \rho_{i}^{B} \] 
These numbers can be used to write a function for the probability that a candidate wins the election:
\[
P^{A} = \sum_{i=1}^{n} \rho_{i}^{A} \ast \left( \sum_{j=0}^{i-1} \rho_{j}^{B} \right) + \frac{1}{2} \left( \sum_{k=0}^{\lfloor n/2 \rfloor} \rho_{k}^{A} \ast \rho_{k}^{B} \right) \] 

(8)
P^{A} is the probability that Candidate A wins the election, which is the probability of receiving more votes than the opponent plus half of the probability of a tie.

3.4 Bringing it Together

The idea is to connect each of the pieces to understand how campaign expenditures affect a candidate’s probability of winning the election. First, there must be a description of the bias of an \( n \)-voter electorate, which can be written as a vector
\[
\vec{V}^{0} = (v_{1}^{0}, v_{2}^{0}, \ldots, v_{n}^{0}) \]
This vector represents the initial distribution of voter political preferences, so \( v_{i}^{0} \) is a real constant and is the initial political preference of voter \( i \). The next important factors are the spending levels of each candidate, \( a \) and \( b \). These values are incorporated into a “shift” function, such as those shown in equations (4), (5), and (6). The shift function gives the change in each voter’s political preference (\( \Delta v_{i} \)) in response to spending levels \( a \) and \( b \). The final political preferences after the changes have taken effect can be written as vector \( \vec{V}^{1} = (v_{1}^{1}, v_{2}^{1}, \ldots, v_{n}^{1}) = (v_{1}^{0} + \Delta v_{1}, v_{2}^{0} + \Delta v_{2}, \ldots, v_{n}^{0} + \Delta v_{n}) \).
These final voter political preferences can be plugged into a “probability of voting” function, such as those in equations (1), (2), and (3). This gives the probabilities of each voter casting his vote for Candidate A \((\alpha_i)\) and Candidate B \((\beta_i)\). Written as a vector, they are \(\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)\) and \(\vec{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)\) respectively. From here, equation (7) calculates the probability that a candidate receives a specific number of votes by aggregating individual probabilities of voting across the electorate. The “probability of winning” function, described in equation (8), can take these probabilities of receiving certain vote totals and determine the probability that a candidate wins the election versus his opponent.

4. Results

Both Snyder and Marsden used functional forms to define the probability of winning an election based on spending levels of the two candidates. These functional forms were somewhat arbitrary, in that they did not take into account the behavior of individual voters. Using a micro-level focus on the behavior of these voters, the analysis in this paper shows that one can see resemblances to both the Marsden model and the Snyder model by using different combinations of parameter values and the functional forms presented in this paper. When considering the effect of spending on probability of winning the election, one might tend to see “S-shaped” curves that begin concave up and come to an inflection point. One might also observe returns to spending that are always decreasing (In other words, the first dollar spent has the highest return to spending).
The existence of an inflection point in the objective function, which differentiates Marsden’s model from Snyder’s, is equivalently the existence of a spending level greater than zero at which marginal return to spending is maximized. The Marsden model is set up so that the highest marginal return to spending occurs when spending levels of the candidates are equal (So, as long as opponent spending is greater than zero, there will exist an inflection point). The model presented in this paper is not designed to be concerned with trying to match an opponents spending level. Then what gives it the similar “S-shape” characteristic? The answer lies in finding the existence of a spending level greater than zero that maximizes marginal return to spending. Where this spending level would occur is dependent upon a combination of the initial distribution of voter political preferences, the spending levels of each candidate, the definition of the shift function, and the definition of the probability of voting function. Each of these factors can play a major role in determining the height and shape of the graph of probability of winning the election on spending.

4.1 Understanding When Marginal Return is Maximized

In order to understand when a functional form similar to those of either Marsden or Snyder might be observed using this paper’s model, it is important to know when return to spending is likely to be maximized. Generally, the greatest changes in probability of winning the election per dollar spent will occur when that next dollar spent causes voters to change their preferences such that there is a large change in their probability of voting. So, ideally for the candidate, that next dollar will reward him a
large shift in each voter’s preferences and a large change in probability of voting per magnitude of shift.

Using a probability of voting function such as that in equation (1a) and (1b) with $c = 0.3$, clearly the largest changes in probability of voting per shift in preferences will be observed when preferences shift close to and through $v_i = 0$ (see Figure 3 and Figure 4). This gives a hint that the largest marginal return to spending might occur at spending levels large enough to cause a shift of some number of voters’ preferences through zero. Suppose the shift function is that of equation (4) with $z = 1$ and $s = 5$. Consider a 3-voter model in which the initial distribution of the voters’ preferences is $\vec{v}_0 = (-2,1,2)$.

*Figure 12* shows Candidate A’s probability of winning as a function of his spending, with Candidate B’s spending held fixed at 2, and *Figure 12A* shows a three dimension version with Candidate B’s spending ($b$) not fixed. The graph strongly resembles a Marsden-like function, as marginal return to spending is increasing until the curve comes to an inflection point. *Figure 13* shows the shift in voter 2’s political preferences, which began at $v_2^0 = .1$. It is no coincidence the spending that would result in a shift of $\Delta v_2 = -.1$, sufficient to shift voter 2’s political preferences to zero, coincides with the spending level at which the slope of the probability of winning graph is maximized (roughly $a = 2.2$). However, as will be seen in the coming examples, seeking out the spending level at which the first voter’s preferences are shifted to $v_i^1 = 0$ is generally not a reliable method of finding the spending level with maximized marginal returns.
4.2 Looking Into Bias, Spending, Shift, and Probability of Voting

Adjusting any of the parameters in this model related to electorate bias, opponent spending, the shift function, or the probability of voting function can have a major impact on determining whether a Marsden or Snyder function will be observed. Keeping all other parameters as they were in the previous example, suppose the initial distribution of voter political preferences changes from $\bar{V}_0^0 = (-2, 1, 2)$ to $\bar{V}_0^0 = (-2, 8, 2)$. This small change gives a graph that now resembles the function of Snyder rather than Marsden, as can be seen in Figure 14. The function is concave down and therefore the highest marginal return to spending is on the first dollar spent, just like in the Snyder model. 

Figure 14A shows that this will also be true while Candidate B spends at least 2 (but not necessarily for $b < 2$, because decreasing opponent spending increases returns to spending through the shift function, which might induce a Marsden example). While Candidate B spends 2, the spending level for Candidate A that would be sufficient to shift the first voter’s preferences to zero is $a = 13.1$. This time, however, no inflection point is observed in the graph. The reason stems from two factors.

The first is the fact that there are decreasing returns to spending in the shift function. This example requires a much larger shift to have voter 2’s political preferences reach zero, where there is the highest sensitivity of probability of voting to change in political preference. For the large shift required here, there is a significantly lower shift per each successive dollar spent. The second reason is that, by the definition of the shift function, voters whose preferences are further from zero shift less than the preferences of
those whose are closer to zero. It is this second factor that is decreasing the return per
dollar spent in the case here, having increased $v^0_2$ to a number further from zero. Either
removing decreasing returns to spending in the shift function or using a less intuitive shift
function that shifts strong preferences the most (such as in equation (6)) would give a
function resembling Marsden here. Candidate B spending has not been changed and
remains at $b = 2$, but it should be noted that, by the definition of the shift function,
increasing Candidate B’s spending decreased return to spending for Candidate A.

Supposing the shift function remains unchanged, a different way to observe
Marsden, building off the most recent example, would be to add copies of each voter.
Now the electorate could be described by $\tilde{V}^0 = (-2,-2,-2, .8, .8, .8, 2, 2, 2)$. If that first
voter’s preferences were to reach zero, so too would the second and third voters’
preferences. This emphasizes the marginal return to spending of the dollar that can
achieve this shift. As expected earlier, the spending level $a = 13.1$, which is sufficient to
shift a voter with $v^0_i = .8$ to the region of $v^1_i = 0$, about marks the inflection point for
this Marsden example (see Figure 15). Adding voters to the electorate does not
necessarily have a large impact on the shape of the “probability of winning” curve. It
does tend to support a Marsden function when multiple voters have identical preferences.
They will respond to campaign expenditures in the same way and are similar to a single
voter with a great deal of voting power. In general, increasing the size of the electorate,
by adding a collection of voters whose preferences are symmetric about zero, will favor
whichever candidate is spending more, as these preferences will be shifted to be
asymmetric for his benefit.
A final adjustment to the model will bring back the Snyder function. Increasing $c$ from .3 to 5 in the probability of voting function makes this happen. The slope has been reduced, which implies that there is less return to strength of political preference in a candidate’s favor (consider flatter slopes in Figure 3 and Figure 4). Shifting a voter’s preference to around $v_i = 0$ is no longer going to have the same strong effect and, in this case, does not induce a Marsden inflection point in the probability of winning function (see Figure 16).

4.3 A Special Case

The model in this paper, given four assumptions, can guarantee a concave down Snyder function. First, every voter initially prefers one candidate ($v_i^0 > 0 \forall i$ or $v_i^0 < 0 \forall i$). Second, the candidate’s opponent spends zero. Third, there are decreasing marginal returns to spending in the shift function. Fourth, there are decreasing marginal returns to strength of political preference in the probability of voting function (see Figure 3 and Figure 4). As an example, consider the initial distribution of voter preferences to be $\tilde{V}^0 = (-.4, -.3, -.2)$. The shift will be defined by equation (4) with $z = 1$ and $s = 5$. The probability of voting will be defined by equations (1a) and (1b) with $c = 5$. Fixing Candidate B’s spending at zero produces the graph shown in Figure 17. Voters’ preferences can only be shifted in the direction of further decreasing marginal returns due to their initial distribution and the fact that the opponent spends zero. This, combined with the decreasing marginal returns to spending in the shift function, always gives a Snyder example.
It is very difficult to make generalized statements about the shape of the probability of winning function, as there are so many tweaks to every scenario that can drastically change the incentives faced by each candidate in how they spend their money. It comes down to the distribution of voters’ political preferences in the electorate, the spending of the opponent, the definition of the shift function, and the definition of the probability of voting function. An important thing to understand is the relevance of the probability of voting function. It dictates the ranges for within which probability of voting is most sensitive to changes in political preference. So much of knowing whether or not there will be a spending level greater than zero at which return to spending is maximized (giving Marsden) depends on how many voters’ preferences are shifting around these ranges. The other parameters are really just determining what it will take to achieve these shifts, but that does have a large impact in determining the shape and height of the probability of winning curve, particularly with the decreasing returns to spending in the shift function.

Using the most intuitive functional forms and parameters does typically result in the observation of a Marsden-like function for describing the probability of winning the election based on spending levels. For the probability of voting function, the most intuitive form featured decreasing returns to strength of political preference in a candidate’s favor, such as in equations (1a) and (1b). For the shift function, the most intuitive form featured the largest shifts in political preference occurring for the most indifferent voters ($v_i$ closest to 0). When a Snyder example is observed, it tends to be due
to rather extreme assumptions, such as very rapidly decreasing marginal returns to spending or an unusually polar initial distribution of voter political preferences.

There is a great deal more than can be done to further investigate micro-level voting models related to popular elections. A possible extension to this research could be to give some utility value $W$ that is received by the winning candidate. Then, one could find the ideal spending levels for each candidate such that spending the next dollar is not worth the marginal increase in probability of winning the election. Another interesting angle to take would be to explore how situations change when campaign spending by each candidate is not simultaneous and there is a shift function such as that in equation (4). For example, if there is only one voter in the electorate and he or she has political preference $V_i = 0$, the candidate who is able to spend first will have the advantage. The candidate who spends second could spend the same amount of money and receive a smaller shift in the voter’s $V_i$. This is because the second candidate’s spending took effect when the voter’s preferences were further from 0 having been already shifted by the first candidate’s spending.
6. Reference List


7. Appendix

**Figure 1** – Snyder’s probability of winning a seat in Congress

- $P(A \text{ wins})$

- $a$

- $0.8$

- $0.6$

- $0.4$

- $0.2$

- $0.0$

- $10$

- $15$

- $20$

- $2$

- $4$

- $6$

- $8$

- $10$

- $P(A \text{ wins})$

- $a$

**Figure 2** – Marsden’s probability of winning a state

- $P(A \text{ wins})$

- $1.0$

- $0.8$

- $0.6$

- $0.4$

- $0.2$

- $0.0$

- $2$

- $4$

- $6$

- $8$

- $10$

- $a$
Figure 3 – Probability of voting for Candidate A (decreasing returns)

Figure 4 – Probability of voting for Candidate B (decreasing returns)
**Figure 5** – Probability for voting for Candidate A (constant returns)

**Figure 6** – Probability of voting for Candidate B (constant returns)
**Figure 7** – Probability of voting for Candidate A (increasing returns)

**Figure 8** – Probability of voting for Candidate B (increasing returns)
Figure 9 – Shift function (indifferent voters shift most)

Figure 10 – Shift function (all voters shift equally)
Figure 11 – Shift function (indifferent voters shift least)

Figure 12 – Graph resembling a Marsden function
Figure 12A – Graph resembling a Marsden function (3D)

Figure 13 – Shift associated with spending
Figure 14 – Graph resembling a Snyder function

Figure 14A – Graph resembling a Snyder function (3D)
Figure 15 – Observing Marsden by duplicating voters

Figure 16 – Back to Snyder by increasing $c$
Figure 17 – Special case (Snyder)