The Merchant’s Dilemma on eBay:
Using Reputational Techniques to Model a Competitive Market

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Abstract

A firm in a competitive environment, that is able to cheat consumers, wishes to maximize profits. This firm has a current incentive to cheat consumers to make additionally revenue. But if the firm builds a reputation for cheating it will not be able to charge as high a price for its product and consumers may avoid it. If firm reputations are sufficiently high there will be an equilibrium where the top firms hold a monopoly over the market and will never be forced to drop out. However, for lower firm reputations, firms are willing to produce in such a way that lower reputation firms may be able to join the cadre of top firms.
1 Introduction

Consider someone with an extensive baseball card collection that contains a large number of cards of various values and qualities, and that they want to sell this collection. She decides that the easiest way to sell her collection is on eBay, or some similar online auction market. Given any one of her cards this seller can tell the truth about its quality, or not. If she lies about the quality of the baseball card, and says that the card is much more valuable than it actually is, she can sell it for a higher price and the consumer who buys this card will have very little legal recourse. Our seller did not have to give her real identity when she made her account on eBay and she may not live in the same state or country as the person to whom she sold the card. Thus, while the buyer could take legal action against her, it is prohibitively expensive and she can feel confident that she will not be sued over the sale.

However, while she may not be strongly legally bound to truthfully reveal the quality of the baseball card she is selling, she may be disciplined by the site’s reputational features. On sites like eBay consumers are able to leave comments on a firm’s page about that firm’s performance. For example, if our seller lied about the quality of a card, then the purchaser of that card would be able to leave a comment warning that our seller lies about the quality of her cards. Future consumers will then see this review and will expect that our seller will cheat them as well. Conversely, if our baseball card seller revealed the true quality of the card that she sold, the purchaser of that card can leave a positive review, suggesting to future consumers that she is a truthful and good seller.

We should also expect that if our seller is known for misrepresenting the quality of the cards that she is selling, consumers will be unwilling to pay as much per card as if she was known to be a good seller. As well, in a market like eBay, there are likely to be other sellers that are willing to sell the same
baseball card as our original seller. These firms also have some reputation for how accurately they represent the cards that they are selling. As such, we should expect that not only are consumers unwilling to pay as high a price to a seller known to cheat, but that they may also be less likely to purchase from that firm at all.

Given this kind of market, this paper addresses the question of when a seller will accurately represent the quality of her product in order to increase her reputation and when she will lie in order to earn larger immediate profits. The paper further addresses the question of which sellers consumers will purchase from.

In order to represent eBay or a similar site, this paper constructs a reputational model under imperfect competition. To model the reputational nature of eBay the model assumes that there a number of firms of two types. One type of firm will always reveal the true quality of their cards; call this the good type. The second type can either reveal the quality of their cards or they can lie to increase their profits; call this the rational type. Each firm also has a reputation which is the consumer’s belief about how likely that firm is a good type. Since reputation on eBay is public, it is assumed that each consumer has the same belief about each firm. As reputation increases, consumers become more and more certain that a firm is of a good type and so consumers will pay a premium to firms with a high reputation because they are less likely to cheat consumers.

This model also features a rate of success. Thus, even if a firm reveals the true quality of the baseball card they are selling, a consumer may still receive a lower quality card than they were expecting. This can be thought of as the probability that the card is injured in the mail, or that the firm thought they were being accurate but unintentionally misvalued the card. This cuts the same way for firms that believe that they are cheating; with some probability
the model assumes that they will actually send a good card by mistake. This assumption prevents consumers from perfectly knowing that a firm is rational after it sends a bad card for the first time.

Finally, to incorporate competition into this market, the model contains a number of firms and a smaller number of consumers. Each firm produces one good and each consumer purchases one good in each period. As such, in the model there exists capacity constraints and a limit on the number of firms who are able to sell their product in each period. There is also no extensive margin. This is partially for modeling tractability, ensuring that firms will be able make a positive profit if they have high reputations, but it also accurately represents online auction markets. We know that products on eBay can take months to sell or may never sell (Livingston, 2004). In addition, firms are able to charge a price for their cards and they feel some cost when they sell a card. This cost can be thought of as the opportunity cost of selling the card on eBay, which is the price that a seller could have gotten for their card from a collectables dealer.

This model is presented in two periods and can be solved through backward induction. Since nothing follows the second period, there is no incentive to build reputation in that period. This period, then, serves to reward firms who build their reputation in the first period. Now consider what happens in the second period: only the firms with the highest reputations are able to sell and each of them makes a profit. This is because a firm with a higher reputation can always offer the same price as one with a lower reputation and still present a better deal to the consumer. As such, in this period there exists a kind of poverty trap that forces lower reputation firms out of the market. Yet, to truly call this a poverty trap we must see not only that low reputation firms cannot produce in the second period, but also that they are unable to build their reputations in the first period in order to move into the group that can produce.
In the first period we find that, except for firms with a very high reputation, it is an equilibrium for firms to produce such that their reputations may drop below the reputations of firms that could not otherwise produce in the second period. We also find that, again, except for very high reputation firms, there are equilibria where higher reputation firms allow lower reputation firms to produce in the hope that they fail to produce a good product and are forced to drop out of the market. As such, this model sets out a range where there is a poverty trap and a range where there is not. If firm reputations are very high then the trap will exist, and if firm reputations are lower it will not. This also means that when firm reputations are high there exists a lower bound on the strategy that a firm can produce with, and so we know that firm reputations must remain above some level in equilibrium.

These results are similar to those found by Livingston (2004) in his analysis of online reputation markets. Livingston created a non-competitive reputational market where consumers were paired with a firm who was either good or bad with some probability. This model only looked at consumer strategy as both good and bad firms played a trivial strategy in every period chosen by nature. Each firm produced one product and consumers decided whether or not they wanted to bid on this product and, if so, how much. This model had two primary results. The first was that firms with a higher reputation had more bidders and were more likely to receive at least one bid for their product. The second was that firms with higher reputations elicited larger bids. As such, high reputation firms were more likely to sell their product and made a larger profit when they did.

In terms of results, Livingston’s paper is very closely aligned with the model presented here. This paper’s model shows that profits are increasing in reputation, and it also shows a limited poverty trap in reputations. This poverty
trap makes it harder, but not impossible, for lower reputation firms to sell their product which is a result similar in spirit to Livingston’s result that higher reputation firms are more likely to receive a bid, and so sell their product. Livingston further supports his claims, and those made by this paper, by analyzing the sales of 861 of a specific brand of gold club. He showed that as reputation increases both prices and the probability of a sale increase. However, he also found that marginal returns to a good signal, review, are decreasing, which is true for the model presented here as well. As a firm’s reputation approaches one or zero good and defective product revelations changes a firm’s reputation less and less.

While the results of this paper are very similar to those found by Livingston, the technique used to find them is more aligned with the literature that analyzes reputation in a competitive market. In this literature, rational firms choose a level of production and the authors look to see when firms will protect their reputation. Generally, they find that when competition is added firms have a greater incentive to build their reputation through not cheating, which can lead to an equilibrium where reputations do not always fall indefinitely over time. For example, Horner (2002) shows an equilibrium whereby some firms will never shirk and others are forced to drop out of the market if their reputation falls below the prior. While they have countered the traditional finding of the literature on reputation, competitive models have generally explored markets where consumer knowledge is severely limited. This is epitomized by Cooper and Ross (1984) who created a model to explore the role of prices in conveying information about product quality. However, to accomplish this, they assume that in the market there are some mass of “uniformed consumers” who calculate the expect utility from buying goods at different price points, but then randomly decide which firm to purchase from at that price. Additionally, Shapiro (1982)
characterizes a market where consumer learning is lagged and Hörner (2002) looks at competition with loyal consumers, consumers who are only able to buy from a single firm and who only receive signals about the quality of that firm. Consumers not tied to a firm can purchase from any firm but only receive price signals. All of these models also assume perfect competition.

This paper separates itself from previous work by making quality signals from firms public, immediate, and costlessly available to consumers. The model thus assumes that consumers on online auction markets always leave feedback about the quality of a sale. Furthermore, this paper also modifies the competitive market to include capacity constraints: that each firm can only produce one good, and it removes the extensive margin: each consumer always purchases one good. While these differences exist, and are substantial, the primary reputation mechanism of this paper is carried over from Hörner (2002) and Mailath and Samuleson (2001). However, these papers assume that there are some firms that will always misrepresent the quality of their baseball cards and others that have a choice whether or not to. The firms that can be truthful, then, want to separate themselves from the ones that can’t. My model though assumes that, instead of bad firms, there are good firms, ones that will always be truthful about their card’s quality, which the rational firms are trying to mimic.

To adhere to the conventions of the literature on reputation, the following model will use the terms producing with high effort and producing with low effort instead of the terms accurately representing the quality of a product and misrepresenting the quality of a product, respectively.

2 The Model

We consider a market where there are \( n \) firms, each of whom has a unique name that carries with it a reputation that follows the firm between periods. Each
of these firms produces a single identical good. There are $k$ consumers where $n > k$. Each of these consumers purchase a single good in each period.

This model contains two types of firms, good types and rational types. Rational types can produce with high or low effort. Producing with low effort increases profits by $S$. A Markov strategy for a rational firm is a mapping

$$
t_i : [0,1]^n \rightarrow [0,1]
$$

where $\tau_i(\phi)$ is the probability that a firm with the $i^{th}$ highest reputation will produce with low effort when the consumers’ prior probability that firm $i$ is good is $\phi_i$. Good firms are only able to produce with high effort and so their strategy is trivial.

All firms face the same cost of production given by $c$ and each sets a price that they sell their good at $p_i$. The profit function for a firm is then:

$$
P_i(\phi) = \begin{cases} 
p_i(\phi) + \tau_i(\phi) S - c & \text{if there is a sale} \\
0 & \text{otherwise}
\end{cases}
$$

If the firm is good then $\tau_i(\phi) = 0$.

The effort level that a firm produces with also effects consumer utility. A firm’s production can lead to the creation and sale of a good product or a defective product. If the product is defective consumers lose some amount of utility $S + \epsilon$, where $\epsilon \geq 0$ is the additional cost that a consumer feels when he receives a defective product beyond the gain to firms. This implies that there is a weak societal loss from cheating, making producing with low effort inefficient. If the product is good there is no utility loss. Effort, however, is not the only element that determines whether consumers receive a good or defective product. With some probability, $\rho$, effort is successful. This means that if a firm produces with high effort then with probability $\rho$, the consumer receives
a good product, and with probability $1 - \rho$ the consumer receives a defective product. Conversely, if a firm produces with low effort, the consumer receives a defective product with probability $\rho$, and a good product with probability $1 - \rho$. It is further assumed that $\rho > 1 - \rho$ such that if a firm produces with high or low effort, they are more likely than not to produce a good or defective product respectively.

Given a firm’s reputation, $\phi_i$, the rate of success, $\rho$, and a firm’s optimal strategy given some reputation level, $\tau_i(\phi)$, a consumer will receive a defective product from firm $i$ with probability:

$$\hat{S}_i(\phi) = (1 - \phi_i) \left[ \rho \tau_i(\phi) + (1 - \rho) (1 - \tau_i(\phi)) \right] + \phi_i \rho$$

Notice that as reputation decreases, $\hat{S}_i(\phi)$ increases for any level of production except for when $\tau_i(\phi) = 0$.

Consumers also receive some utility, $\mu$, from their purchase and pay $p_i$, the price being charged by whichever firm they purchase from. Thus, a consumer’s utility from purchasing a product from firm $i$ is:

$$C_i(\phi) = \mu - \hat{S}_i(\phi) (S + \varepsilon) - p_i(\phi)$$

In each period the sequence of events, shown in Figure 1, is as follows. At the beginning of period $t$ each firm has some reputation $\phi_{ti}$ and sets the price $p_{ti}$ at which it will sell its good. Each consumer then purchases from some firm, as there is no extensive margin. Rational type firms that have been purchased from then make an unobserved effort choice. All firms that have been purchased from then produce a good and receive revenue $P_{ti}$. This revenue is not affected by firm type, effort level, or the realization of the good. The quality of each good produced is then revealed to all firms and consumers, and all agents use Bayes’
rule to update their beliefs about the types of firms. Importantly, all agents receive the same signal about each firm regardless of which firm the consumer purchased from. Finally, each firm chooses whether it stays in the market, and firms that leave are replaced by firms with reputations randomly drawn from some distribution.

This paper looks for a symmetric Markov perfect equilibrium where, firms maximize expected profits, consumers’ maximize expected utility, consumers’ expectations are correct, and consumers use Bayes’ rule to update their beliefs. In such an equilibrium, consumers know the strategy played by each kind of firm given any reputation level, and so a firm’s strategy uniquely determines the equilibrium updating rule that the consumers use. A Markov perfect equilibrium also implies that the equilibrium is not history dependent. Thus, the only relevant information to an agent is the vector of reputations; how any particular reputation came to be is not informative. Thus, history-dependent equilibria, such as punishment strategies, are not considered. Further, symmetry implies that firms with the same reputation amongst the same market of other firms
will exert the same effort level and set the same price.

3 Equilibrium Analysis

For tractability this model is solved in two periods and using backward induction. The equilibrium of the final period is presented first.

Equilibrium in the Second Period (without Reputational Concerns)

Each firm will enter the second period with a commonly known reputation, $\phi_2i$. Since the second period is the last period, these firms will have no incentive to protect or build their reputations and will simply seek to maximize within period profits.

It is important to mention here that there are two ways that a firm can send a signal about its reputation. The first way, which was mentioned above, is the revelation of a firm’s product being either good or defective. The second way is through the price that a firm sets. If a rational firm sets a price that a good firm would not set then consumers will know that that firm is rational. The same applies for good firms setting a price that a rational firm would not set. Due to this ability to signal types with prices, firms may have an incentive to set different prices than they otherwise would to distinguish themselves or blend in. This is a price pooling equilibrium.

Proposition 1. Suppose there are $n$ sellers with reputations $\phi_2i, \in \{\phi_21, ..., \phi_2n\}$ where WLOG $\phi_2i \geq \phi_2i+1$, and further assume that these equalities are strict.

1. In any equilibrium, it must be that firm $i$ where $i \in \{1, ..., k\}$ sells its product and make positive profits and firm $j$ where $j \in \{k + 1, ..., n\}$ do not sell and make 0 profits.
(1.2) If \((\rho - \phi_{2i})(S + \epsilon) > c\), then there is a price pooling equilibrium such that \(p_{2i} > c\) for \(i \in \{1, ..., k + 1\}\).

(1.3) If \((\rho \phi_{2,k+1} + (1 - \rho)\phi_{2,k+1})(S + \epsilon) < c\), then in any equilibrium, for \(i \in \{1, ..., k\}\) firm \(i\) sets price \(p_i = (\hat{S}_{2,k+1} - \hat{S}_{2i})(S + \epsilon) + p_{2,k+1}\) and makes profit \(P_{2i} = (\hat{S}_{2,k+1} - \hat{S}_{2i})(S + \epsilon) + S\).

All proofs are in appendix A.

(1.1) We can imagine a number of firms on eBay all selling some baseball card. These firms all have some history on the site and have collected a number of reviews each, positive or negative. Now, let’s say that all of these firms set the same price. A consumer looking through this market would clearly purchase from the firm with the highest reputation, as the consumer prefers the smaller chance of being cheated. In the same way we can consider what would occur in the model if we assumed that a firm that was not among the \(k\) with the highest reputation sold its product in the final period, call this firm “firm \(j\)”. This would mean that there was a firm in the \(k\) highest that was not selling its good, call this firm “firm \(i\)”. However, firm \(i\) could always set its price to that of firm \(j\) and produce more consumer surplus than firm \(j\) because \(\hat{S}_{2i} > \hat{S}_{2j}\) by the nature of their reputations. Thus, it would be able to sell its good and would make a positive profit doing so.

However, in each period firms choose their production strategy along with their price. Thus, given firm \(i\) choosing some production strategy \(\tau_{2i}(\phi)\) it is imaginable that firm \(j\) would be able to choose a \(\tau_{2j}(\phi)\) such that \(\hat{S}_{2j} > \hat{S}_{2i}\). This is to say that firm \(j\) may be able to produce in such a way that consumers are actually less likely to receive a defective product from it than they are from firm \(i\) even though firm \(i\) has a higher reputation. This would allow firm \(j\) to produce instead of firm \(i\) and make positive profits in this period. Unfortunately for firm \(j\) this strategy is prevented by the commitment problem.
in this period.

This model assumes that production takes place after a purchase is made. Then, because this is the final period if a firm has been purchased from it maximizes in period, and so total profits, by setting $\tau_2(\phi) = 1$ and so producing with low effort with probability 1. Further, this model does not incorporate a method that would allow a firm to credibly commit to producing with a certain effort level before a purchase is made. As such, while a firm may promise to produce with a certain effort level in this period it is merely posturing, and consumers will not treat it as informative.

We can also look at this commitment problem in the context of eBay. Let’s take our original seller, who has a large baseball card collection, and say that she puts them all up for sale at once. Say that the market for each card is self contained, and that she has different relative reputations in each market, that for some she is in the top and in others she is not. Her cards will sell slowly and her reputation will change as they do. We can imagine her changing her prices along the way to account for her new reputation and the way that she moves around in the markets. Now say that she has sold all of her cards except for one, and that consumers can see this, say that they check her profile and notice that this is her only card for sale. After she has sold this last card she is always better off taking the money from the sale and then selling the card to a collectables dealer since her reputation is no longer valuable to her. This however, would cause consumers to never buy her last card, and then the card before that and before that and so on, but this problem is mitigated on eBay by consumers not knowing how much stock a seller has left.

(1.2) To see the intuition behind this result consider a consumer on eBay who is looking for a valuable baseball card, and imagine that this consumer has some idea about the true value of the card he is looking for, say he has shopped
around at collectable card shops to discover this value. Then, among the cards on eBay that the consumer sees, there is one that is being sold for a price that is significantly below the true value of the card. Upon seeing this, the consumer must either believe that the firm that is selling this card does not know the true value of the card, that the seller is in a huge hurry to sell the card, or that the card is not of the value professed. Since a firm can easily look at the prices being charged by other firms selling a similar product, it is clear that the firm could not have been unclear about the card’s value. Also, if a firm was eager to sell their card they could always have taken it to a pawn shop or collectables store for a price slightly under the card’s value. Thus, a consumer has to assume that the card that is being sold is not actually of the value that the firm claims that it is, and the consumer then can be very certain that if he buys this very low priced card that he will be cheated. As such, he will avoid it. Then, since firms know that consumers will avoid cards if they are set below a certain price these firms will always set their prices at or above the true value of the card.

There is an exception to this equilibrium though, and it applies to firms that already have an extremely low reputation. Let us again consider a consumer, except in this instance she sees a firm with a large number of negative reviews and very few or no positive reviews. This consumer may assign some probability to the possibility that she will receive a high quality card from this firm however, it is very low. In this case the firm can better compete with other firms by revealing that it is lying, and will send a lower quality card than promised, which frees it to set a lower price which may entice the consumer, even though the card is of poor quality.

We must here also consider off equilibrium path beliefs that consumers may have. There exists many other equilibrium consumer beliefs about the price that a good firm will charge, such that $p_{2i} \geq c$. For example, consumers may
believe that good firms will only charge a price such that \( p_{2i} > c + \gamma \), where \( \gamma > 0 \). This belief is an equilibrium if, given these beliefs, good firms will only charge a price that is above \( c + \gamma \). We can think about a firm that sees a market where it knows that it will not be purchased from unless it sets its price high enough. Of course, the firm will comply with this belief in order to sell its good. However, these beliefs do not separate firms, since good firms and rational firms are equally able to fulfill the consumer’s beliefs. As such, if a consumer sees a firm offering a price slightly below \( c + \gamma \) but above \( c \) it should still expect that this firm may be good, which implies that consumers should change their beliefs about the prices that a good firm would charge. This should occur until consumers believe only that good firms will always charge a price such that \( p_{2i} \geq c \). As this belief does separate firms as a good firm will not charge a price below this point.

(1.3) This term implies that the revenue that a firm is able to make is determined by firm \( k + 1 \). This is seen in that the price that firm \( i \) charges is the price that firm \( k + 1 \) charges plus the additional benefit to consumers from firm \( i \) being less likely to send a defective product. This result is due to the capacity constraints and the lack of an extensive margin, both of which prevent the firms with \( k \) highest reputation, that sell, from competing with each other. In the model, providing a large consumer surplus relative to another firm that is able to sell does not increase the probability of a sale. The only factor which effects whether a firm will sell its good is whether it provides more consumer surplus than the first firm that cannot sell, which is firm \( k + 1 \). As such, each firm needs only provide an amount of consumer surplus above the amount that firm \( k + 1 \) can to sell, and then the firm maximizes its profits by taking the rest of the surplus for itself.

We can think about a firm that has a large number of baseball cards to sell
but only one Babe Ruth card, and that this firm has developed a high reputation through selling off some of these other cards. Further, this firm knows that there is a relatively contained market for Babe Ruth cards and that in some period of time, say a week, there is a relatively constant number of these cards that are sold, let’s say five. Thus, if a firm wants to sell this baseball card within a week it knows that it only has to be among the week’s five best deals. Further, it is wasteful for this firm to try to make its particular Babe Ruth card the best deal of the week because it would have sold regardless, and the drop in price necessary to make it the best deal is lost profit.

**Equilibrium in the First Period (with Reputational Concerns)**

Each firm will enter the first period with a commonly known reputation, \( \phi_i \), chosen by nature. In this period the firm’s incentive is to maximize lifetime profits i.e. first period profits and the discounted sum of the second period’s profits.

For tractability the first period game is solved for two firms which don’t have the same reputation. Without loss of generality we assume that firm 1 has a higher reputation than firm 2, such that \( \phi_{1,1} > \phi_{1,2} \). We assume further that there is one consumer, such that only one of the firms is able to sell in this period.

While a simpler two player game is used in this period primarily for tractability its intuitions should largely apply to other market sizes as well. This is due to the importance of firm \( k+1 \) in the analysis of the second period. In that period we saw that each of the top \( k \) firms competed only with firm \( k+1 \); as such to find the profits of those top firms we needed only know the reputation of firm \( k+1 \). As well, firms with reputations below firm \( k+1 \) were found to be entirely irrelevant to the model’s equilibrium. As such, in solving the two player game firm 2 becomes firm \( k+1 \) and firm 1 stands in for firm \( i \) for \( i \in \{1,\ldots,k\} \). We
should then be able to add firms either above or below firm 2 without changing the analysis.

**Proposition 2.** Suppose there are 2 sellers with reputations $\phi_i \in \{\phi_1, \phi_2\}$ where WLOG $\phi_1 \geq \phi_2$, and further assume that this equality is strict. Further, there exist $\phi_L, \phi_H \in (0,1)$ with $\phi_H > \phi_L$.

(2.1) When reputation is within some range, such that $\phi_H > \phi_i > \phi_L$, both firms will produce with a mixed strategy such that $\tau_{1i} \neq 1$ and $\tau_{1i} \neq 0$, when $\phi_i \geq \phi_H$ or $\phi_L \geq \phi_i$ firms prefer to always produce with low effort such that $\tau_{1i} = 1$.

(2.2) There is an equilibrium where firm 1 produces such that $\varphi^D(\phi_{1,1}) < \phi_{1,2}$ when $\phi_{1,2} = \frac{(1-\phi_1)\rho}{(1-\phi_1)\rho + \phi_1(1-\rho)}$ and $\varphi^{G^*}(\phi_{1,1}) > \phi_{1,1} > \varphi^{G^*}(\phi_{1,1}) - \frac{(1-\rho)\varphi^{G^*}(\phi_{1,1})}{\rho - (1-\rho)\varphi^{G^*}(\phi_{1,1})}$ where

$$
\varphi^{G^*}(\phi_{1,1}) = -2\delta\rho S + \delta S + \sqrt{\delta(2\rho - 1)^2 S(\delta S + 2\rho + 2\epsilon)}
\delta(2\rho - 1)^2(S + \epsilon)
$$

(2.3) In any equilibrium, firm 2 will produce in the first period when $\varphi^{G^*}(\phi_{1,2}) > \phi_{1,2} > \frac{(1-\rho)\varphi^{G^*}(\phi_{1,2})}{\rho - (1-\rho)\varphi^{G^*}(\phi_{1,2})}$, $\phi_{1,1} < \frac{\phi_{1,2} \rho}{(1-\phi_1)\rho + \phi_1(1-\rho)}$, and

$$
G_2 \frac{S}{2\rho - 1} - \tau_{1,2}(\phi)S + D_2 \delta \phi_{1,1}(2\rho - 1)(S + \epsilon) + S + (\hat{S}_2(\phi) - \hat{S}_1(\phi))(S + \epsilon) + \delta[\varphi(\phi_{1,1}^G)]G_1(\frac{1}{2\delta}\varphi(\phi_{1,1}^G)(2\rho - 1)(S + \epsilon) + S) > D_2 \delta \phi_{1,1}(\frac{1}{2}\phi_{1,1}(2\rho - 1)(S + \epsilon) + S)
$$

when $\varphi^D(\phi_{1,1}) > \phi_{1,1}$ or

$$
G_2 \frac{S}{2\rho - 1} - \tau_{1,2}(\phi)S + (\hat{S}_2(\phi) - \hat{S}_1(\phi))(S + \epsilon) + \delta[\varphi(\phi_{1,1}^G)]G_1(\frac{1}{2\delta}\varphi(\phi_{1,1}^G)(2\rho - 1)(S + \epsilon) + S) + \varphi(\phi_{1,1}^D)D_1(\frac{1}{2}\varphi(\phi_{1,1}^D)(2\rho - 1)(S + \epsilon) + S) > D_2 \delta \phi_{1,1}(\frac{1}{2}\phi_{1,1}(2\rho - 1)(S + \epsilon) + S)
$$

when $\phi_{1,2} > \varphi^D(\phi_{1,1})$. Otherwise firm 1 will produce.

All proofs are in appendix A.

(2.1) Let’s first consider what a mixed strategy means in terms of sellers on eBay. One way of picturing this is to imagine a firm putting a single baseball card up for sale. If the firm were to play a mixed strategy then it would randomly
decide if it is going to lie about the quality of the card or tell the truth, by
flipping a coin maybe. Another, more intuitive way of thinking about how a
firm would produce with a mixed strategy is if it put a large number of cards
up for sale at once. The firm, in this case may decide to lie on some proportion
of the listing and tell the truth on the others.

Now we consider why a firm might want to mix between high and low effort.
In the model firms play a mixed strategies in order to change consumer beliefs
given the result of effort. If a rational firm produces with only high effort
it becomes equally likely that a rational firm and a good firm would produce
a defective product. As such, the reveal of a defective or good product will
not change consumer beliefs about the firm’s type. Now imagine that a firm
produces with a mix of high and low effort putting more weight on producing
with high effort. In this case a reveal will change a firm’s reputation, but because
the likelihood of a good result for a rational firm is not much less than for a
good firm and the likelihood of a defective result is not much more, a firm’s
reputation will not change by much given either reveal. The more likely a firm
is to produce with low effort the more a firm’s reputation will change given
either reveal. As such, firms want to balance the amount that there reputation
will change with the gains from cheating in the first period.

On eBay we can also imagine that firms are trying to manage there reputations to get the maximum value out of them. A firm on eBay wants a high
enough reputation that it is able to provide a deal that is good enough that
consumers will purchase its card. It also wants the additional profits from selling each card that it receives due to the higher prices the firm can charge when
its reputation is higher. However, the marginal results of reputation are de-
creasing. As our firm keeps accurately representing its cards its number of good
comments will keep going up. However, if our firm has, say, a hundred positive
comments, consumers may not take notice of a couple more while our firm will have missed out a significant profit to get those couple more. As such, we should see our firm trying to balance keeping its reputation high and making a profit off of that reputation by cheating, through accurately representing some cards and lying about the others. This is the essence of a mixed strategy.

This equilibrium only holds within a certain range of reputations though. Outside of that range there is always an equilibrium where the firm only cheats. This is due to the decreasing effect of a signal on reputation when reputations approach 0 or 1. When a firm’s reputation is very high consumers are so sure of the firm’s type that a product revelation will change its reputation very little either up or down. This is also true for very low reputation firms. As such, reputational gains and losses are most salient for middling reputation firms. When reputation change is small the profit difference for either reveal is also small. Given this, when the difference between the profits that reputational reveals imply is less than the gain from cheating, then the firm prefers to always cheat. This means that for certain reputations firms don’t care to build or maintain their reputations, though if reputations dropped enough, or gained enough, firms would begin to groom their reputations again.
This range of reputations is shown in Figure 2 where the area between the two lines is the range of reputations for which reputation is valuable given a change in one variable. Notice for $\delta$ reputation is only ever valuable after $\delta \approx .52$. The ranges for changes in $\rho$ and $\epsilon$ both follow the basic shape of that for $\delta$.

To see this intuitively, consider a firm on eBay. This firm has a massive number of good reviews and little to no bad reviews. Thus, consumers are very sure that this firm will send the quality of card that it promised. Now consider what would happen if this firm sent a lower quality card and received a poor review. The number of good reviews would still be so overwhelming that consumer’s opinions about the firm would barely change, the bad result could be written off as a fluke. Then, of course, if the firm got one more good review that would have even less of an effect on consumer expectations. Thus, since the change in reputation from another sale by this firm will have almost no effect on its reputation it is clearly best off misrepresenting its next card and sending a lower value card to receive the extra profits from that sale. We can also quickly consider a terrible firm. It has an overwhelming number of bad reviews and consumers are essentially sure that it will send a lower quality card than promised, and even with a couple more positive reviews consumers would not change their minds. This firm then, assuming it is in a market bad enough that it is able to sell a card at all, may as well continue cheating.

(2.2) This proposition states when firm 1 will produce in such a way that if it produces a defective product its reputation will fall below firm 2’s reputation. This would mean that firm 1 would drop out of the market after period 1, since it would be unable to sell in the second period, and firm 2 would move on to the second period instead. For firm 1 to produce in this way two conditions must be met. The first condition is that it is possible for firm 1 to produce in such a way that its reputation falls below that of firm 2. It may be that firm 1’s
reputation is so much larger than firm 2’s, that even with the largest possible reputation drop firm 1 would remain above firm 2. The second condition is that firms 1’s level of mixing must be an equilibrium.

We see in Figure 3 that there is, for reasonable values of the variables, a range of reputations for which it is an equilibrium for firm 1 to produce in such a way that it might not produce in the second period. Figure 3 shows this range between the two lines and for a range of values of $\epsilon$. The range for other variables follow the same pattern as for $\epsilon$. Further, using a simulation we can show that the bottom section always falls underneath $\phi_L$ meaning that whenever reputation is so low that a firm will no longer wish to produce such
that its reputation will fall below its competition’s reputation if it has fallen into the range where it is an equilibrium for the firm to always cheat, such that \( \tau_{i1}(\phi) = 1 \). Thus, that bottom section disappears, since always cheating must result in firm 1’s level of reputation dropping below firm 2’s if any level of mixing would, thus if the first condition that we outlined above stills holds.

As such, it will only not be an equilibrium for firm 1 to produce such that it may fall below firm 2 if its reputation is high enough, specifically if \( \phi_{1,1} > \varphi^G_*(\phi_{1,1}) \). This result has a very clear intuition behind it. When \( \phi_{1,1} > \varphi^G_*(\phi_{1,1}) \) firm 1 has so much expected profit in the second period that it will not risk being forced out of the market. We can also think about this in terms of eBay.

Consider a firm on eBay, as this firm accumulates good reviews, and its reputation grows, it is able to make more and more money off of each baseball card that it sells, even when it doesn’t cheat. Thus, as the firm’s reputation grows it will be willing to spend more resources to be able to stay in the market. As such, if there are a number of high reputation firms that control the market for certain cards, our firm has a greater interest in staying among these firms as its reputation grows. It will then be less willing to cheat to the extent that it would no longer be able to sell its cards. Simply put a firm in a better position will be less willing to risk that position. This is a very intuitive result.

(2.3) This proposition outlines the two conditions that must be met for firm 2 to produce in the first period instead of firm 1. The first condition states that there must be an equilibrium where firm 2 will produce, which only occurs in the range set out in proposition (2.2). The second condition states that it must be more profitable for firm 1 to not produce in the first period than it would be to produce in the first period. Let us consider the second condition first.

It may be that in the first period firm 2 is willing to take a loss on its
product in order to gain a reputational advantage and so be able to make a positive profit in the second period that makes up for this loss. If firm 2 sets its price low enough then, even with its reputational advantage, firm 1 would also have to take a loss in the first period if it were to produce. Further, if that loss is big enough it may be in firm 1’s best interest to not produce in the first period and instead count on firm 2 failing to produce a good product, which allows firm 1 to continue into the next period. However, further intuition, past seeing that there are circumstances where firm 1 prefers to let firm 2 produce in the second period, is muddied by the term’s complexity.

However, we can still imagine this result on eBay. Consider a market for some baseball card where there a number of very high reputation firms who control the market. Now consider a new firm who enters the market. This firm will struggle to sell its cards due to the reputational advantage of its competitors, even if it sets its price at the cost of the card. Due to its comparatively low reputation the other firms are still offering a better deal. This firm could wait for one of the top firms to leave the market or it could set its price below the value of the card, and possibly even lower than that. As this firm doesn’t only have to set its price low enough that it is a better deal than its competitors but also low enough that its competitors don’t want to bother competing with it. However, the model shows that this behavior is possible and can be profitable.

Now let us consider our second condition, that there must be an equilibrium where firm 2 wants to produce. This condition is primarily a fiction of the model that results from the definition of a mixed strategy equilibrium, however, it is an intuitive result if we take it to eBay. The intuition is very similar to that found in proposition (2.2) in which we saw that a firm in a better position is less willing to cheat to such an extant that it is no longer competitive in the market. Instead of competing though we find that a firm in a better position is
less willing to allow its competition to gain an advantage over it.

Let us consider a number of high reputation firms that have cornered the market for some baseball card. The higher the reputations of these firms the greater a profit they are making off of each sale. Now, let’s say that a new firm enters the market, regularly it would be unable to sell. However, this firm is willing to take a loss so that it can build its reputation. The higher the reputations of the firms already in the market the more important it is that this new firm not be able to build its reputation, as it could diminish the profits of the older firms, or start pushing these firms out of the market entirely. Thus, as reputations increase it becomes increasingly advantageous for these top firms to take a loss on some sales in order to keep the market free of entrants and protect their monopoly.

4 Conclusion

Sellers on eBay face a dilemma in dealing with their customers. These sellers are able to cheat their consumers by misrepresenting the value of what they are selling. This cheating generates additional short-term revenue. However, sellers also care about what consumers think of them. If consumers think that they are unlikely to cheat, sellers find it easier to sell their goods and they are able to sell their goods for a higher price, increasing long term profits. This paper uses reputational techniques within a competitive model to simulate how firms act given these incentives.

In this model, a firm must choose a level of production that balances the short term profit of cheating with the longer term benefits of a good reputation, which are a greater ability to sell a good and the ability to charge a higher price for the good when it is sold. This model finds three main results. The first is that in the second period only the firms with the highest reputation are able to
sell their goods. The second is that in the first period there exists an equilibrium where a higher reputation firm produces with an effort level such that it may be forced out of the market if it produces a defective product. Finally, the model finds that, also in the first period, there exists an equilibrium where a lower reputation firm is able to sell its good to try to build its reputation. However, the final two results do not hold when reputation is high enough that the top firms make a profit sufficiently high that they jealously guard their spot in the market.

This model was unable to answer the question of how firm reputations change in equilibrium. Whether the reputations of rational firms will approach zero in equilibrium or whether the addition of competition would bound reputations to some higher value. As such, one extension to this paper would be to redo this model as an infinite horizon game in order to observe how reputations change.

Appendix A

The Second Period

In the second period, suppose there are $n$ sellers, indexed by $i \in \{1, \ldots, n\}$, with reputation $\phi_i$ (where reputation is the belief of being the good type). WLOG suppose $\phi_1 \geq \phi_2 \geq \ldots \geq \phi_n$ and, for simplicity, assume each of these inequalities is strict. Let $\phi = (\phi_1, \ldots, \phi_n)$ denote the vector of reputations.

There are $k$ buyers in the market, each purchasing a single good. Let $\tau_{2i} : [0,1]^n \rightarrow [0,1]$, where $\tau_{2i}(\phi)$ denotes the probability that the agent with $i^{th}$ highest reputation chooses low effort, conditional upon remaining in the market. Let $\hat{S}_{2i}(\phi)$ denote the probability that the good produced by the firm with the $i^{th}$ highest reputation will be defective. we have:

$$\hat{S}_{2i}(\phi) = (1 - \phi_i) \left[ \rho \tau_{2i}(\phi) + (1 - \rho) (1 - \tau_{2i}(\phi)) \right] + \phi_i \rho$$
Let $i < j$, so that $\phi_i > \phi_j$. Notice that $\hat{S}_{2i} (\phi) < \hat{S}_{2j} (\phi)$ provided that:

\[ \frac{1 - \phi_i}{1 - \phi_j} < \frac{\tau_{2i}(\phi)}{\tau_{2j}(\phi)}, \]

which will hold as long as $\tau_{2i}$ is not too much larger than $\tau_{2j}$.

The higher reputation firm can ‘maintain’ its reputational advantage so long as it does not shirk too much, relative to a lower reputation firm.

Let $p_{2i} (\phi)$ denote the second period price charged by the firm with the $i^{th}$ highest reputation. The profit of such a firm is:

\[
P_{2i} (\phi) = \begin{cases} 
p_{2i} (\phi) + \tau_{2i} (\phi) S - c & \text{if there is a sale} \\
0 & \text{otherwise} \end{cases}
\]

A consumer’s expected utility from purchasing the good from firm $i$ is:

\[
C_{2i} (\phi) = \mu - \hat{S}_{2i} (\phi) (S + \varepsilon) - p_{2i} (\phi)
\]

**Proof of Proposition 1.1**

The consumers will purchase the $k$ goods with the highest expected utilities. Let $\bar{C} (\phi)$ be the consumer’s equilibrium expected utility from purchasing the $k^{th}$ most desirable good. Then if good $i$ is amongst the $k$ most desirable goods, $p_{2i} \leq \mu - \hat{S}_{2i} (\phi) (S + \varepsilon) - \bar{C} (\phi)$.

**Lemma 3.** In any equilibrium, it must be that $C_{2i} \geq \bar{C} (\phi) = C_{2,k+1} (\phi)$ for all $i \in 1, \ldots, k$ and $C_{2,j} (\phi) \leq \bar{C} (\phi)$ for all $j \in k+2, \ldots, n$.

**Proof.** The proof is in two steps. First, I show that if $i < j$, then $C_{2i} (\phi) \geq \bar{C} (\phi)$ whenever $C_{2j} (\phi) \geq \bar{C} (\phi)$. To see this, suppose not — i.e. suppose $C_{2j} (\phi) \geq \bar{C} (\phi) > C_{2i} (\phi)$. Then $P_{2i} (\phi) = 0$. But there is a feasible strategy for firm $i$ that generates positive profits. (For example, if firm $i$ chooses the same policy as firm $j$, then $\hat{S}_{2i} < \hat{S}_{2j}$, and so $C_{2i} > C_{2j}$.) Hence, firm $i$ must be selling its good. This implies that firms $1, \ldots, k$ must all successfully trade, which ensures that $C_{2i} (\phi) \geq \bar{C} (\phi)$ for all $i \in 1, \ldots, k$. Now, since the remaining firms cannot
trade, it must be that these firms had worse consumer surplus than $\bar{C}(\phi)$. \qed

**Proof of Proposition 1.2**

*Proof.* This proof is in two steps. The first part follows from profit condition of a good type firm. Again to see this, suppose not and suppose that firm $i$ is of a good type — i.e. $c > p_{2i}$. In this case firm $i$ is making negative profits and can beneficially deviate by not selling a good and setting profits to 0. The second part shows that $p_{2i} \geq c$ for a rational firm and follows from consumer expectations. To see this, suppose not — i.e. $c > p_{2i}$. By the first part of our proof consumers know that a good type firm will not set prices this low. Thus, consumers will then rationally set $\phi_i = 0$ such that firm $i$ is set to the lowest, $n^{th}$, reputation slot. Hence, by proposition 1.1 firm $i$ will no longer be able to sell its good and its profits will be 0. This however, does not hold when reputation is so low that the reputation decrease is so slight the consumer surplus can be increased even with the drop. \qed

**Proof of Proposition 1.3**

This proof uses three helping Lemmas. We first show that firm $i$ for all $i \in 1,..,k$ provides consumer surplus equal to the surplus provided by firm $k + 1$. We then show that firms must always produce with low effort. We then use that result to show the level of consumer surplus supplied by firm $k + 1$. Finally, we combine these to solve for the price and profit of firm $i$ for all $i \in 1,..,k$.

**Lemma 4.** *In any equilibrium, it must be that $C_{2i} = \bar{C}(\phi)$ for all $i \in 1,..,k$.***

*Proof.* To see this, suppose not — i.e. suppose $C_{2i} > \bar{C}(\phi)$, as we know from proposition 1.1 that there is always a strategy for firm $i$ that generates $C_{2i} \geq \bar{C}(\phi)$. Then, firm $i$ can increase $p_{2i}$ by some amount $\epsilon$ such that $C_{2i} - \epsilon \geq \bar{C}(\phi)$. Such a price increase ensures firm $i$’s good will still be bought and increases profits by $p_{2i}$. This does not depend on the firm’s type. \qed
Lemma 5. In any equilibrium, it must be that $\tau_{2i}(\phi) = 1$ for all $i \in 1, \ldots, n$.

Proof. This is given by the equilibrium to the production subgame, shown in Figure 1. To see this, assume not — i.e. $1 > \tau_{2i}(\phi)$. However, if a firm is purchased from they can increase $P_{2i}$ by increasing $\tau_{2i}(\phi)$ some amount $\epsilon$ such that $1 \geq \tau_{2i}(\phi) - \epsilon$ without fear of punishment, as production happens last within the game and the second period is the last period. 

Lemma 6. In any equilibrium, it must be that $\bar{C}(\phi) = \mu - \hat{S}_{2,k+1}(\phi)(S + \epsilon) - c$ where $\hat{S}_{2,k+1}(\phi) = (1 - \phi_{2,k+1})(1 - \rho) + \phi_{2,k+1}(\rho)$ when $(\rho - \phi_{2i})(S + \epsilon) > c$

Proof. This proof is in two steps. First I show that $p_{2,k+1} = c$. To see this, suppose not — i.e. $p_{2,k+1} > c$, we know that $p_{2,k+1} \geq c$ by proposition 1.2. But firm $k+1$ can increase $C_{2,k+1}(\phi)$ while retaining $P_{2,k+1} \geq 0$ by decreasing $p_{2,k+1}$ by some amount $\epsilon$ such that $p_{2,k+1} - \epsilon \geq c$. The second part follows from lemma 5 which tells us that all firms must set $\tau_{2i}(\phi) = 1$ which implies $\hat{S}_{2,k+1}(\phi) = (1 - \phi_{2,k+1})(1 - \rho) + \phi_{2,k+1}(\rho)$. 

Proof. We can know use our lemmas to find the price and profit of firm $i$ for $i \in 1, \ldots, k$. Price follows from lemmas 4 and 6. Which together give us that:

$$\mu - \hat{S}_{2i}(\phi)(S + \epsilon) - p_{2i} = \mu - \hat{S}_{2,k+1}(\phi)(S + \epsilon) - c$$

$$p_{2i} = (\hat{S}_{2,k+1}(\phi) - \hat{S}_{2i}(\phi))(S + \epsilon) + c$$

From lemma 5 we further know that $\tau_{2i}(\phi) = 1$. Substituting price and strategy into our profit function we get:

$$P_{2i} = (\hat{S}_{2,k+1}(\phi) - \hat{S}_{2i}(\phi))(S + \epsilon) + S$$
**First Period**

In the first period there are again \( n \) sellers, specified identically to the sellers of the second period. There are also \( k \) buyers in the market, who again purchase a single good each. Let \( \tau_{1i} : [0,1]^n \rightarrow [0,1] \), where \( \tau_{1i}(\phi) \) denotes the probability that the agent with \( i^{th} \) highest reputation chooses low effort. Let \( \hat{S}_{1i}(\phi) \) denote the probability that the good produced by the firm with the \( i^{th} \) highest reputation will be defective. We have:

\[
\hat{S}_{1i}(\phi) = (1 - \phi_i) \left[ \rho \tau_{1i}(\phi) + (1 - \rho) (1 - \tau_{1i}(\phi)) \right] + \phi_i \rho
\]

Let \( p_{1i}(\phi) \) denote the second period price charged by the firm with the \( i^{th} \) highest reputation. The profit of such a firm is:

\[
P_{1i}(\phi) = \begin{cases} 
p_{1i}(\phi) + \tau_{1i}(\phi) S - c & \text{if there is a sale} \\
0 & \text{otherwise}
\end{cases}
\]

A consumer’s expected utility from purchasing the good from firm \( i \) is:

\[
C_{1i}(\phi) = \mu - \hat{S}_{1i}(\phi) (S + \epsilon) - p_{1i}(\phi)
\]

Notice that \( \hat{S}_i \), the consumers expectation that a good has a defect, \( P_i \), the profit of firm \( i \), and \( C_i \), a consumers expected utility remain constant between periods.

Additionally let \( \varphi(\phi_{1i}) \), the second period reputation of firm \( i \), be:
ϕ(φ_{1i}) = \begin{cases} 
\frac{\phi_{1i}}{1-\phi_{1i}} \left( 1 - \tau_{1i}(\phi) \right) (1 - \rho) + \phi_{1i} \rho & \text{if the product is good} \\
\frac{\phi_{1i}(1-\rho)}{(1-\phi_{1i})[(1-\rho)\tau_{1i}(\phi) + (1-\rho)(1-\tau_{1i}(\phi)) + \phi_{1i}(1-\rho)]} & \text{if the product is defective} \\
\phi_{1i} & \text{if there is no sale} 
\end{cases}

For short hand let us say that ϕ(φ_{1i}^G) is the second period reputation of a firm if the product is good and ϕ(φ_{1i}^D) is the second period reputation of a firm if the product is defective. Also notice that a firm’s reputation does not change unless a consumer purchases from it, because production does not occur and so no consumer can receive a signal about the firm.

Let δ be the discount factor between periods and, for simplicity, let us assume that the reputation of replacement firms is drawn from the uniform distribution U(0, 1).

**The Two Player Game**

For tractability we solve for the two player game. In this market let us assume that there is only one consumer. As well let us specify the two firms indexed by i ∈ {1, 2} where φ_1 > φ_2.

Before we show the proofs to Proposition 2 we some aspects of the model that will be used throughout those proofs. We first show the expected profit of the firm that continues into the second period in that period.

To do this we first solve for the expected value of the reputation of the entrant, φ_{2e}. The probability that φ_{2i} > φ_{2e} is given by φ_{2i}. Where if φ_{2e} > φ_{2i}, P_i = 0 as the entrant sells instead. Further we know that the expected value of the entrant’s reputation E(φ_e) = \frac{φ_e}{2} given that φ_1 > φ_e. Thus we can now solve for (\hat{S}_{2e}(φ) - \hat{S}_{2i}(φ)):
\[
\hat{S}_{2e}(\phi) - \hat{S}_{2,1} = \left[(1 - \frac{1}{2}\phi_{2i}) + \frac{1}{2}\phi_{2i}\rho\right] - \left[(1 - \phi_{2,1}) + \phi_{2,1}\rho\right] = \frac{1}{2}\phi_{2i}\rho - \frac{1}{2}\phi_{2i}(1 - \rho) = \frac{1}{2}\phi_{2i}(2\rho - 1)
\]

Substituting this into our expression for firm 1’s expected profit in the second period we have:

\[
P_{2i} = \frac{1}{2}\phi_{2i}(2\rho - 1)(S + \varepsilon) + S
\]

We now define three cases in which a firm’s profits may fall. This first case is only possible for firm 1. In the first case \(\varphi(\phi_{1,1}^D) > \phi_2\), meaning that firm 1’s reputation is high enough that even if it’s product is defective it’s reputation will remain higher than firm 2’s. In this case firm two will always drop out after the first period and be replaced by a firm with reputation \(\phi_e\) drawn randomly from \(U(0, 1)\). First though, for notation we define \(G = (1 - \tau_{1,1}(\phi))\rho + \tau_{1,1}(1 - \rho)\) as the chance that a product is good given that a firm is rational and \(D = \tau_{1,1}(\phi)\rho + (1 - \tau_{1,1})(1 - \rho)\) as the probability that a product is defective given that a firm is rational. Thus, firm 1’s expected lifetime profits in the first case

\[
P_1^1 = P_{1,1} + \delta[\varphi(\phi_{1,1}^G)G_1(\frac{1}{2}\varphi(\phi_{1,1}^G)(2\rho - 1)(S + \varepsilon) + S) + \varphi(\phi_{1,1}^D)D_1(\frac{1}{2}\varphi(\phi_{1,1}^D)(2\rho - 1)(S + \varepsilon) + S)]
\]

In the second case \(\phi_2 > \varphi(\phi_{1,1}^D)\) meaning that if firm 1’s product is revealed to be defective then firm 1’s reputation will fall below that of firm 2. Thus, if firm 1’s product is revealed to be defective it will not be able to trade in the second period and will drop out of the market at the end of the first period.
This is also the expected profit if firm 2 produces in this period, as it will only produce in such a way that it is able to pass firm 1 and so be able to produce in the second period. We can write the lifetime expected profit of a firm in this case as:

\[ P_i^2 = P_i + \delta[\varphi(\phi_i)G_i\left(\frac{1}{2}\varphi(\phi_i)(2\rho - 1)(S + \epsilon) + S\right)] \]

Notice that the only change in the expected lifetime profits of firm 1 between the two cases is the omission of the final term which stands for the expected profits of firm 1 in the second period if its good is revealed to be defective. This term is simply set to 0.

The third case is also only possible for firm 1. It is the expected profits of firm 1 if it does not produce:

\[ P^0_i = D_2\delta\phi_{1,1}(\frac{1}{2}\phi_{1,1}(2\rho - 1)(S + \epsilon) + S) \]

It is also important to note that for many combinations of \(\phi_{1,1}\) and \(\phi_{2,1}\) there are some \(\tau_{1,1}(\phi)\) for which case one is generated and some for which case two is generated. To solve for when cases will be generated it is first important to show that \(\varphi(\phi_{1,1})\) is decreasing in \(\tau_{1,1}(\phi)\). We can see this by taking the derivative of \(\varphi(\phi_{1,1})\):

\[ \frac{d}{d\tau_{1,1}(\phi)}\varphi(\phi_{1,1}) = -\frac{(\rho - 1)(2\rho - 1)(\phi_{1,1} - 1)\phi_{1,1}}{(\rho[2\tau_{1,1}(\phi)(\phi_{1,1} - 1) + 1] + \tau_{1,1}(\phi)(-\phi_{1,1}) + \tau_{1,1}(\phi) - 1)^2} \]

Given any legal values of \(\rho, \phi_{1,1}\), and \(\tau_{1,1}(\phi)\), \(\frac{d}{d\tau_{1,1}(\phi)}\varphi(\phi_{1,1})\) is always negative. This means that as a firm plays low effort with a higher probability the firms reputation drops further if a defective product is received. Using this we can solve for the reputation difference that will lead to case 1 regardless of what
strategy is played by firm 1. We know that the greatest reputation drop is created by maximizing $t_{1,1}(\phi) = 1$. Putting this into $\varphi(\phi^D_{1i})$, we get that the most a firm's reputation can fall is by, and so the highest firm 2's reputation can be relative to firm 1's to guarantee that case 1 occurs is:

$$\Delta \varphi(\phi^D_{1i}) = \frac{(1 - \phi_{1,1})\rho}{(1 - \phi_{1,1})\rho + \phi_{1,1}(1 - \rho)}$$

If firm 2's reputation is any higher than this than there exists some $\tau_{1,1}(\phi) > \bar{\tau}_{1,1}(\phi)$ which will generate case 2 and some $\tau_{1,1}(\phi) \leq \bar{\tau}_{1,1}(\phi)$ which will lead to case 1. We know this because firm 1 can always produce with $\tau_{1,1}(\phi) = 0$ in order to maintain their current reputation, as:

$$\varphi(\phi^D_{1i}) = 1 - \frac{(1 - \phi_{1i})((\rho)0 + (1 - \rho)(1 - 0))}{(1 - \phi_{1i})(\rho)0 + (1 - \rho)(1 - 0) + \phi_{1i}(1 - \rho)}$$

$$= 1 - \frac{(1 - \phi_{1i})(1 - \rho)}{(1 - \rho)}$$

$$= \phi_{1i}$$

**Proof of Proposition 2.1**

This proposition is solved using two helping lemmas. The first shows that firms must not produce with $\tau_{1i}(\phi) = 0$ and the second shows that within the range $\phi_H > \phi_{1i} > \phi_L$ firms must not produce with $\tau_{1i}(\phi) = 1$.

**Lemma 7.** In any equilibrium, it cannot be that $\tau_{1i}(\phi) = 0$ for all $i \in 1, 2$.

**Proof.** Given that consumers believe that firm $i$ produces with $\tau_{1i}(\phi) = 0$ we know that $\varphi(\phi^D_{1i}) = \varphi(\phi^G_{1i}) = \phi_{1i}$. Hence, a firm is not punished if consumers receive a defective product or rewarded if the consumer receives a good product. Thus, a firm prefers to produce with $\tau_{1i}(\phi) = 1$ which generates profits $S$ and has no reputational cost, thus increasing lifetime expected profits $P_i$ by $S$. □
We define $\phi_L$ and $\phi_H$ such that when a firm has reputation $\phi_L \geq \phi_i$ or $\phi_H \geq \phi_i$ there exists an equilibrium where firm $i$ produces with $\tau_{1i}(\phi) = 1$. This occurs when firms produce with $\tau_{1i}(\phi) = 1$ and:

$$(2\rho - 1) \delta \varphi(\phi_{1,1}^D)(\frac{1}{2} \varphi(\phi_{1,1}^D)(2\rho - 1)(S + \varepsilon) + S) \geq (2\rho - 1) \delta \varphi(\phi_{1,1}^G)(\frac{1}{2} \varphi(\phi_{1,1}^G)(2\rho - 1)(S + \varepsilon) + S)$$

Noting that in some cases $\varphi(\phi_{1,1}^D) = 0$. This expression implies that if a firm’s reputation is high or low enough then the reputational changes from a good or bad outcome are less than the gains from cheating. In these cases reputation is not valuable and so will not be built or kept up.

**Lemma 8.** In any equilibrium, when $\phi_H > \phi_i > \phi_L$ it cannot be that $\tau_{1i}(\phi) = 1$ for all $i \in 1, 2$.

*Proof.** This follows from the definition of $\phi_H$ and $\phi_L$. Within this range firm $i$ prefers to produce with the $\tau_{1i}(\phi) = 0$ as the reputational gains are greater than the gains from cheating.\[\square\]

Taken together lemmas 7 and 8 imply that within the range $\phi_H > \phi_i > \phi_L$ firm $i$ prefers to mix between cheating and not cheating.

**Proof of Proposition 2.2**

*Proof.** Then, from the definition of a mixed strategy equilibrium, given some level of mixing, $\tau_{1i}(\phi)$, firm $i$ must be indifferent between cheating and not cheating. To find this condition we set equal to each other the expected profits of cheating and not cheating given consumer beliefs that a firm is producing with some $\tau_{1i}(\phi)$ and we get:

$$(2\rho - 1) \delta \varphi(\phi_{1,1}^D)(\frac{1}{2} \varphi(\phi_{1,1}^D)(2\rho - 1)(S + \varepsilon) + S) = (2\rho - 1) \delta \varphi(\phi_{1,1}^G)(\frac{1}{2} \varphi(\phi_{1,1}^G)(2\rho - 1)(S + \varepsilon) + S)$$
However, when $\phi_{1,1}$ is close enough to $\phi_{1,2}$ firm $i$ can mix between high and low effort in such a way that if its product was revealed to be defective then its reputation would drop below the reputation of firm 2. To be more precise for some values of $\tau_{1,1}(\phi)$, $\phi_{1,2} > \varphi(\phi_{1,1}^D)$. This means that firm 1 will be unable to produce in period 2 and will drop out of the market at the end of period 1. Thus, for these values of $\tau_{1,1}(\phi)$, given a defective revelation firm 1 will earn 0 profits in the second period. We can then input this into our indifference condition by setting $\varphi(\phi_{1,1}^D) = 0$, because the $\varphi(\phi_{1,1}^D)$ represents the chance of production in the second period which is 0 for firm 1 in this situation. Substituting this into our indifference condition we get:

$$S = (2\rho - 1)\delta\varphi(\phi_{1,1}^G)(\frac{1}{2}\varphi(\phi_{1,1}^G)(2\rho - 1)(S + \varepsilon) + S)$$

As the left side of our equation is static we can trivially solve for the $\varphi_{1,1}^G$ that will lead to this equilibrium:

$$\varphi_{1,1}^G = \frac{-2\delta\rho S + \delta S \pm \sqrt{\delta(2\rho - 1)^2 S(\delta S + 2S + 2\varepsilon)}}{\delta(2\rho - 1)^2 (S + \varepsilon)}$$

Therefore we know that when firm 1’s reputation is above this level firm 1 must always produce and firm 2 must never produce. We can also solve for the lower bound on the firm 1s reputation such that it is an equilibrium to produce such that $\varphi_{1,1}^D < \phi_{1,2}$. To do this we use the maximum that a reputation can change which results from our updating equations when $\tau_{1,1}(\phi) = 1$ which is:

$$\Delta\varphi_{1,1}^G = \frac{\phi_{1,1}\rho}{(1 - \phi_{1,1})(1 - \rho) + \phi_{1,1}\rho}$$

We then solve for period 1 reputation and get
This gives us a range in which producing in such a way that firm 1 would drop below firm 2’s reputation is an equilibrium which is \( \phi_{1,1}^* > \frac{(1 - \rho)G^*(\phi_{1,1})}{\rho - (1 - 2\rho)G^*(\phi_{1,1})} \).

**Proof of Proposition 2.3**

*Proof.* We know from proposition 1.1 that firm 1 could always produce in the first period by nature of its higher reputation. However, it may be beneficial not to. Firm 1 will prefer to produce in the first period when its profits are greater when it does so, specifically when \( P_{1,1}^1 > P_{1,1}^0 \) or when \( P_{1,1}^2 > P_{1,1}^0 \), with these terms defined above in our discussion on profit cases, depending on the equilibrium strategy of firm 1. Oppositely firm 1 will let firm 2 produce in the opposite case, when \( P_{1,1}^0 > P_{1,1}^1 \) or when \( P_{1,1}^0 > P_{1,1}^2 \). To specify these terms we can be more specific about \( P_{1,1} \) which is dependent on the maximum consumer surplus that firm 2 is willing to provide. To construct this we must determine firm 2’s expected second profits when it produces, which are given by \( P_{1,2}^2 \), as firm 2 always exists in the second case as is stated above. Given this and the mixing indifference condition we have:

\[
S = (2\rho - 1)\delta\varphi(\phi_{1,1}^G)(\frac{1}{2}\varphi(\phi_{1,1}^G)(2\rho - 1)(S + \epsilon) + S)
\]

Which comes from the proof of proposition 2.2. We can then substitute this term into \( P_{1,2}^2 \) to give us:

\[
P_2 = P_{1,2}^2 + G_2 \frac{S}{2\rho - 1}
\]

Where \( G \) is given by firm 2’s equilibrium strategy. Expanding \( P_{1,2} \) we have:
\[ P_2 = p_{1,2} + \tau_{1,2}(\phi)S - c + G_2 \frac{S}{2\rho - 1} \]

Where \( \tau_{1,2}(\phi) \) is firm 2’s strategy which is given by the indifference condition. To determine the maximum amount of consumer surplus that firm 2 is willing to provide we must also construct firm 2’s expected lifetime profits if it doesn’t produce. If firm 1 produces in such a way that \( \phi^D(\phi_{1,1}) > \phi_{1,2} \) then \( P_2 = 0 \). However, if firm 1 produces in such a way that \( \phi_{1,2} > \phi^D(\phi_{1,1}) \) its lifetime expected profits are given by:

\[ P_2 = D_1 \delta \phi_{1,2}(\frac{1}{2} \phi_{1,2}(2\rho - 1)(S + \epsilon) + S) \]

Which is the same as firm 1’s profit from not producing in the first period, where \( D_1 \) is given by firm 1’s optimal strategy. Further, firm 2 is willing to provide up to its expected gain from producing as consumer surplus. To see this, suppose not and for simplicity call the expected value of producing \( x \) and the expected value of not producing as \( y \). If \( x > y \) and firm 2 is providing some amount of consumer surplus \( C_2 \) and firm 1 is providing some amount of consumer surplus, \( C_1 \), such that \( x - y > C_1 > C_2 \). Firm 2 is making some expected profit \( y \). However, firm 2 can increase \( C_2 \) by \( z \) such that \( x - y \geq C_2 + z \) and make profits \( x - C_2 + z > y \). Given this we can solve for \( p_{1,2} \):

\[ p_{1,2} = c - G_2 \frac{S}{2\rho - 1} - \tau_{1,2}(\phi)S + D_1 \delta \phi_{1,2}(\frac{1}{2} \phi_{1,2}(2\rho - 1)(S + \epsilon) + S) \]

Where the final term is 0 if firm 2 produces in such a way that \( \phi^D(\phi_{1,1}) > \phi_{1,2} \). This produces consumer surplus:
\[ C_2 = \mu + c - G_2 \frac{S}{2\rho - 1} - \tau_{1,2}(\phi)S + D_1 \delta \phi_1,2(\frac{1}{2} \phi_1,2(2\rho - 1)(S + \epsilon) + S) - \hat{S}_2(\phi)(S + \epsilon) \]

To match this consumer surplus firm 1 must provide price:

\[ p_{1,1} = c - G_2 \frac{S}{2\rho - 1} - \tau_{1,2}(\phi)S + D_1 \delta \phi_1,2(\frac{1}{2} \phi_1,2(2\rho - 1)(S + \epsilon) + S) + (\hat{S}_2(\phi) - \hat{S}_1(\phi))(S + \epsilon) \]

By the logic of lemma 4 from period 2 firm 1 maximizes profits by setting price no lower than this. Using this price we can construct the profits of firm 1 if it produces in the first period if it produces such that \( \varphi^D(\phi_{1,1}) > \phi_{1,2} \):

\[ P_1 = G_2 \frac{S}{2\rho - 1} - \tau_{1,2}(\phi)S + D_1 \delta \phi_1,2(\frac{1}{2} \phi_1,2(2\rho - 1)(S + \epsilon) + S) + (\hat{S}_2(\phi) - \hat{S}_1(\phi))(S + \epsilon) + \delta[\varphi(\phi_{1,1}^G)G_1(\frac{1}{2} \varphi(\phi_{1,1}^G)(2\rho - 1)(S + \epsilon) + S)] \]

We can also construct the profits of firm 1 if it produces such that \( \phi_{1,2} > \varphi^D(\phi_{1,1}) \):

\[ P_1 = G_2 \frac{S}{2\rho - 1} - \tau_{1,2}(\phi)S + (\hat{S}_2(\phi) - \hat{S}_1(\phi))(S + \epsilon) + \delta[\varphi(\phi_{1,1}^G)G_1(\frac{1}{2} \varphi(\phi_{1,1}^G)(2\rho - 1)(S + \epsilon) + S)] \]

Given the profits firm 1 receives when it does not produce we can say that firm 1 will produce, it produces such that when \( \varphi^D(\phi_{1,1}) > \phi_{1,2} \):

\[ G_2 \frac{S}{2\rho - 1} - \tau_{1,2}(\phi)S + D_1 \delta \phi_1,2(\frac{1}{2} \phi_1,2(2\rho - 1)(S + \epsilon) + S) + (\hat{S}_2(\phi) - \hat{S}_1(\phi))(S + \epsilon) + \delta[\varphi(\phi_{1,1}^G)G_1(\frac{1}{2} \varphi(\phi_{1,1}^G)(2\rho - 1)(S + \epsilon) + S)] > D_2 \delta \phi_{1,1}(\frac{1}{2} \phi_{1,1}(2\rho - 1)(S + \epsilon) + S) \]

We can further say when firm 1 will produce if it produces such that \( \phi_{1,2} > \varphi^D(\phi_{1,1}) \):

\[ G_2 \frac{S}{2\rho - 1} - \tau_{1,2}(\phi)S + (\hat{S}_2(\phi) - \hat{S}_1(\phi))(S + \epsilon) + \delta[\varphi(\phi_{1,1}^G)G_1(\frac{1}{2} \varphi(\phi_{1,1}^G)(2\rho - 1)(S + \epsilon) + S)] > D_2 \delta \phi_{1,1}(\frac{1}{2} \phi_{1,1}(2\rho - 1)(S + \epsilon) + S) \]

Further by the proof of proposition 2.2, we know that for firm 2 to produce
it must be that:

$$\varphi^G(\phi_{1,2}) > \phi_{1,2} > \frac{(1 - \rho)\varphi^G(\phi_{1,2})}{\rho - (1 - 2\rho)\varphi^G(\phi_{1,2})}$$

Or there will be no equilibrium where firm 2 is able to produce. □

References


