Winning the Electoral College:
How Presidential Candidates Optimally Allocate Resources across States

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Abstract:

This paper addresses the question of how candidates in a two-player, \( n \)-state election can optimally allocate their resources across states to maximize their probability of winning. It begins with an overview of the Electoral College system and an analysis of past theoretical papers. The model in this paper diverges from this literature in two respects. First, candidates are assumed to maximize their probability of winning rather than their expected number of electoral votes. Many authors have assumed the latter to simplify the problem, but the two objectives are not necessarily equivalent. Second, the functional form analyzed, which maps spending within a state to the probability that a candidate wins the state, has yet to be investigated in detail. In a simplified set-up no pure-strategy equilibrium emerges in this model, but a mixed strategy equilibrium does exist. This breaks with previous authors such as Snyder (1989), who have shown the existence of a pure-strategy equilibrium under certain conditions. The crucial difference between Snyder’s model and the one examined in this paper is the concavity of each candidate’s objective function. Concavity arises in Snyder’s model and fails to emerge in this paper because of differences in the second derivatives of the functions for the probability of winning a state.

This analysis offers many avenues for further research, such as expanding the results to include more states with different numbers of electoral votes, candidates with unequal resources, and bias within states. Some hypotheses and ideas for unequal endowments and bias are discussed in the concluding sections.
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1. Introduction

There is little doubt that campaign spending plays an important role in determining election outcomes. If campaign advertisements did not positively affect voters, we would not be bombarded with the steady stream of political messages prior to elections. In the last presidential election cycle, George W. Bush and John Kerry raised over six hundred million dollars combined, a significant amount of which was spent on advertising. If the spending in this year’s primaries is any indication, the upcoming 2008 campaigns may set new records. A challenge facing candidates every cycle is where to spend this money. For example, is it optimal for a candidate to focus the majority of his resources on a few key states or employ a broader, but thinner strategy? This paper seeks to address this problem by examining how candidates with a fixed allocation of resources in a two candidate, $n$ state contest should distribute these funds across states to maximize their chances of winning the election.

Our analysis diverges from previous literature in two respects. First, we consider a different functional form for the mapping of spending within a state to the probabilities of winning that state. In addition, we assume that candidates attempt to maximize their probability of winning the election, whereas many previous authors make the assumption that candidates try to maximize their expected number of electoral votes. With these assumptions we demonstrate the dramatic impact small shifts in one player’s strategy can have on his opponent’s optimal allocation. In addition, we argue that under a simplified set-up, no pure-strategy equilibrium occurs in this game, although a mixed strategy equilibrium does exist. This is at odds with other authors’ results, which show that a pure-strategy equilibrium can exist under certain conditions. Finally, we explore some possibilities to expand the model, such as allowing candidates to have unequal endowments of resources and including biases within states.

The argument begins with a general overview of the American Presidential Election system and the Electoral College. Section 3 covers results from past literature and highlights some of their weaknesses. Section 4 introduces the model and our

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assumptions, and discusses the objective function in more detail. Section 5 describes results and possible extensions. The concluding section summarizes our analysis.

2. Electoral College System

The basic structure of the American Presidential Election system was stipulated by Article II of the Constitution and modified by the 12th Amendment. In this contest many candidates choose between numerous strategies in their competition for electoral votes. There are two major candidates in each election, a Democrat and a Republican. Many minor candidates also compete, but their impact is generally minimal. Winning the majority of electoral votes virtually guarantees winning the Electoral College and thereby becoming president. There are a total of 538 votes at stake. Each state receives one vote for each of its House seats and two additional votes for its two Senate seats. Washington D.C. also has three votes. Since over eighty percent of electoral votes are tied to House seats, which are approximately proportional to population, the number of votes a state receives is roughly proportional to its population. The most populous state, California, currently has 55 electoral votes while many smaller states, such as Delaware, only have 3. The distribution is modified after each census (in line with the reallocation of House seats) to reflect shifts in population across states.

The Constitution grants states the right to decide how they allocate their electoral votes, but almost all award them to the candidate who receives the most popular votes within the state. The two exceptions, Nebraska and Maine, allocate a portion of their electoral votes to the winner of each of their respective congressional districts. However, the overall winner in each state has also won each of the districts in the presidential elections since these states have shifted to this method.

When polls close across the country, the candidate with the majority of the Electoral Votes (at least 270) is declared the winner. In the event of a tie in the Electoral College (or if neither candidate receives a majority, which is not an issue if third parties do not receive votes), the election is decided in the House of Representatives. However, the likelihood of each candidate receiving exactly 269 votes is quite small. In addition, the House vote would likely be along party lines and the result could be predetermined.
This reduces the necessary number of electoral votes for the candidate who has the support of the House to 269.

3. Review of Past Literature

Over the past 50 years, numerous economists, political scientists and mathematicians have explored variations of the problem confronting candidates in an $n$ state election of how to optimally divide their resources across states. Two main branches of literature have emerged. The first analyzes how resources have been spent in the past and specific strategies candidates have employed. The second tackles the problem from a theoretical perspective and tries to determine what drives spending decisions in a general framework. The later is more closely related to the objective of this paper, but the more historical literature is worth a brief introduction.

Shaw (2006), who worked closely with George W. Bush in the 2000 and 2004 elections, has an interesting insider’s perspective of how presidential candidates construct their strategies. He emphasizes the familiar division of states into five specific categories based on previous election results and other political factors: “safe democratic”, “leans democrat”, “battleground”, “leans republican”, and “safe republican.” However, grouping or ranking the states by their competitiveness does not yield an optimal allocation strategy. Candidates must condense these categories and decide where they plan to spend. Shaw describes four possible methods that a candidate could employ, and gives examples of campaigns that have utilized them. The first method is an “offensive” strategy, where a candidate pours his resources into close states and those where his opponent is slightly favored, a tactic the Bush campaign used successfully in 2000. The counter strategy is “defensive,” in which a candidate devotes resources to battleground states and those where he is slightly ahead. There is also a more balanced or “mixed” strategy where a candidate focuses most of his money on “toss-up” states, but also allocates some to states in which one candidate is slightly favored. Shaw claims that this is one of the more

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popular strategies utilized by candidates.\textsuperscript{3} In addition, he states, “I leaned toward a mixed strategy [in 2000] simply because it allowed us to hedge our bets – we could always become more aggressive if conditions continued to look propitious.”\textsuperscript{4} Finally, a candidate could use a “high-risk” strategy and pour all of his resources into the “battleground” states. Gore’s 2000 strategy is an example of this tactic. Shaw concludes by emphasizing that candidates are “rational” because they attempt to maximize their probability of winning the election.\textsuperscript{5} He stated this same conclusion based on his analysis of theoretical papers during an initial presentation to Karl Rove and other top Bush advisors in the summer of 1999.\textsuperscript{6}

Two important issues run throughout this theoretical literature. First, there is some disagreement about or at least unwillingness to commit to a functional form for the probability that a candidate wins a given state (see Section 4.1 for a more detailed discussion). Some authors have defined a particular function, while others have merely suggested possibilities. There is also some disparity amongst author’s definitions of candidate objectives. Much of the early literature makes the simplifying assumption that candidates try to maximize their expected number of electoral votes rather than their probability of winning the Electoral College. Although the optimal strategies for these objectives are identical under certain conditions, it is questionable whether these hold in reality (see Section 4.2 for a further explanation). Based on how they address these questions and other assumptions, some authors are able to show the existence of a pure-strategy equilibrium in their models under certain conditions.

In one of the first prominent articles on the subject, Brams and Davis (1974) argue that candidates should allocate their resources across states, such that the allocations are proportional to the electoral votes of each state raised to the $3/2$’s power. They derive this result by first assuming that candidates are maximizing their expected number of electoral votes, although they admit this might not be the correct definition of candidate objectives.\textsuperscript{7} The authors proceed to argue that there appears to be empirical

\textsuperscript{3} ibid, p. 53.
\textsuperscript{4} ibid, p. 54.
\textsuperscript{5} ibid, p. 67.
\textsuperscript{6} ibid, p. 45.
evidence that candidates match each other’s resource allocations across states. They claim that an equilibrium exists at this point. However, at this point candidates only locally maximize their objective functions so this is not a true pure-strategy equilibrium. Nevertheless, the further assumption that the number of uncommitted voters in a state is proportional to its number of electoral votes yields the “3/2’s rule.” Finally, the authors show empirical support for this result and try to explain why this large state bias exists.

The flaws in Brams and Davis’s argument are highlighted in Colantoni, Levesque and Ordeshook (1975). One major problem is the fragileness of the local equilibrium where candidates match each other’s allocations. Brams and Davis admit, “in fact, it can be shown that when a candidate knows the allocations of his opponent under the Electoral College system, the most devastating strategy that he can generally use against him is to spend nothing in the smallest state and instead use these extra resources… to outspend his opponent by a slight amount in each of the other states.”8 In other words, simply matching the opponent’s strategy is far from optimal. In addition to this criticism, Colantoni, Levesque and Ordeshook also seek alternative explanations for the empirical evidence supporting the “3/2’s rule” cited by Brams and Davis. They argue, “the apparent empirical support…can be attributed, first, to the correlation between the size and competitiveness of states and to the failure of candidates to visit states.”9 One key problem is that Brams and Davis ignore the issue of bias. Colantoni, Levesque and Ordeshook write, “Candidates should be concerned not only with size but also with the likelihood that resources can swing a state from one candidate to another.”10 In other words, competitiveness is important. They proceed to show that size and competitiveness are empirically related, thereby partly explaining the large state bias in election spending. Finally, they note that Brams and Davis ignore corner solutions in their problem where candidates spend zero resources in at least one state, which could also help clarify the tendency of candidates to spend more in larger states.

Some of the more recent literature focuses on the objective of maximizing the probability of winning. Snyder (1989) analyzes the resource allocation problem in the

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8 ibid, p. 123.
10 ibid, p. 144.
framework of a congressional election. Congressional elections mirror presidential elections except that the players are political parties and each state (or district) has the equivalent of one electoral vote. Snyder considers both the problem of a party trying to maximize the number of seats won in Congress as well as the objective of trying to win a majority of seats. For the purposes of this paper, where we assume candidates are trying to maximize their chances of winning the election, the objective of trying to gain a majority in Congress is more relevant.

Snyder claims that with a specific functional form it is possible to show that a unique pure-strategy equilibrium exists in the game with the objectives of trying to win a majority of seats. In this equilibrium candidates spend on each state (or district). After establishing the existence of an equilibrium he proceeds to show that spending will tend to be higher in districts where the party leading the overall election is ahead. This result stems from how pivotal a district is, where pivotal is defined as the probability that “winning or losing [the district] would make the difference between winning or losing a majority of the legislature.” However, Snyder’s results rest on his specification of the functional form mapping the amounts of money candidates spend within a state to the probability that they win the state. This is discussed in more detail in the next two sections.

Strömberg (2002) addresses this objective of maximizing the chances of winning in a presidential election framework. In his main argument he treats campaign endowments as time for appearances rather than financial resources. However, these different endowments lead to similar problems, as each candidate is dealing with some limited number of resources, which he can employ to positively swing voters. Strömberg does include a section on advertising expenditures, but candidates’ strategies are based on media markets rather than states.

Many of Strömberg’s main conclusions are in line with Snyder’s. He argues that there are two important factors which determine resource allocation to a state: how close the vote in the state is predicted to be and how likely it is that winning or losing the state could change the outcome of the election. States which have high probabilities of both

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are referred to as “decisive swing states.” Strömberg concludes that candidates ought to allocate more time to states where the candidate who is leading nationally is ahead, which mirrors Snyder’s conclusion that candidates allocate more resources to districts where the leading party is ahead.

Strömberg arrives at his results by first focusing on the cumulative distribution of voters within each state. He then derives an objective function for a candidate attempting to maximize their probability of winning, but writes, “It is difficult to find strategies which maximize the expectation of the above probability of winning.” In order to proceed he reduces the objective function to a form which boils down to a candidate trying to maximize their expected number of electoral votes. This paper attempts to analyze a similar problem without using this simplification.

4. Model

In a general framework, candidates compete in an election across \( n \) states: 
\( (1, \ldots, k, \ldots, n) \). In a presidential election \( n = 51 \). Each state has a given number of electoral votes, \( e_k \). The total number of electoral votes is \( \sum_{k=1}^{n} e_k = E \).

For simplicity, we assume two players compete in this game, Candidate A and Candidate B. Although third party candidates also participate, they rarely come close to winning electoral votes and they spend far less than the two major party candidates. In the months prior to an election, the two contenders scramble across the country attempting to amass the resources to sustain their bid. Although this process continues throughout the campaign, we assume that candidates have accumulated a fixed amount of resources, which they can draw upon to support their candidacy. For \( I \in [A, B] \), \( R_I \) represents Candidate I’s endowment of resources.

Given this endowment the candidates much choose how to distribute it across states. The decision is actually much more complicated as there are hundreds of ways to allocate the resources within states, such as through TV advertising or campaign mailings,

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13 ibid, p. 5.
but to simplify the problem we assume that spending within each state is homogenous. (For a discussion of how strategies might differ for various types of resources see Bartels (1985)). In addition, we make the simplifying assumption following the previous literature that there are no spillover effects from spending. The probability that a candidate wins a state depends exclusively on the money the candidate and his opponent allocate to that state, not allocations to any neighboring states. In reality, when candidates spend in certain media markets on the peripheries of states, such as Philadelphia, the positive impact on voters may spread beyond states lines.

We also make the assumption that the probability of winning a state is statistically independent of the probability of winning any other state. In other words, winning one state has no effect on whether or not a candidate wins another state. This is useful because it means the probability that a candidate wins two particular states is simply the product of the probabilities that the candidate wins each individual state. Although this is a helpful assumption, it also may not hold in reality. For example, if one demographic of partisan voters turns out strongly across the country, their impact jointly affects the probability of winning in many states.

Given these assumptions, we define $r_k^i \in R_+$ as the total resources Candidate $I$ allocates to state $k$. Candidate $I$’s strategy set consists of all $n$ dimensional vectors, \{r_1^i, r_2^i, ..., r_n^i\}, such that his total spending does not exceed his total endowment:

$$S_I = \{\{r_1^i, r_2^i, ..., r_n^i\} \in R^n_+ \text{ s.t. } r_1^i + r_2^i + ... + r_n^i \leq R_I\}.$$

### 4.1 Function Describing the Probability of Winning a State

As discussed in the previous literature section the functional form mapping spending within a state to the probability of winning the state plays a crucial role in this problem. We define the probability that Candidate A wins state $k$ as:

$$P_k (r_a^k, r_b^k) = \frac{1}{1 + e^{-(r_a^k - r_b^k)}}$$

Some authors have mentioned probability of winning functions based on this functional form (see for instance Colantoni, Levesque and Ordeshook (1975)), but so far
it has not been investigated in detail. The following graph is an example of this function where $r^b_k = 5$.

![Graph showing Candidate A’s probability of winning state $k$ when Candidate B spends 5 in the state.](image)

Some important properties are worth noting. First, $P^a_k (r^a_k, r^b_k) \in (0, 1)$ and $P^a_k$ is continuous for all $r^a_k, r^b_k$. Second, if Candidate A matches Candidate B, both candidates have a 50% chance of winning. More generally, the closer Candidate A comes to matching his opponent, the closer the race is within the state (both candidates have probabilities of winning near 0.5). Furthermore, the marginal return to spending for Candidate A, $\frac{\partial P^a_k}{\partial r^a_k}$, is greatest when the candidates match each other. In general, the closer the election is within a state, the more impact additional spending has for Candidate A. The return to spending is also symmetric around $r^b_k$ and positive for any value of $r^a_k$: $\frac{\partial P^a_k}{\partial r^a_k} \mid_{r^a_k = r^b_k + \Delta} = \frac{\partial P^a_k}{\partial r^a_k} \mid_{r^a_k = r^b_k - \Delta} > 0$. The absolute value of the change in the return to spending for Candidate A is also symmetric about $r^b_k$:

$$\frac{\partial^2 P^a_k}{\partial (r^a_k)^2} \Big|_{r^a_k = r^b_k + \Delta} = \frac{\partial^2 P^a_k}{\partial (r^a_k)^2} \Big|_{r^a_k = r^b_k - \Delta}.$$

The return to spending increases as Candidate A catches up to Candidate B’s allocation,
but declines as soon as Candidate A exceeds his opponent’s spending. The change in the 
return is exactly zero when candidates match each other: \[ \frac{\partial^2 P_k^a}{\partial (r_k^a)^2} \bigg|_{r_k^a=r_k^b} = 0. \] Finally, this 
function is symmetric in the sense that all of these properties also hold for Candidate B’s 
probability of winning function,

\[ P_k^b(r_k^a, r_k^b) = \frac{1}{1 + e^{-(r_k^a - r_k^b)}} = 1 - \frac{1}{1 + e^{-(r_k^a - r_k^b)}} = 1 - P_k^a(r_k^a, r_k^b), \] treating \( r_k^a \) as fixed.

Snyder uses a different functional form, which he derives from many previous 
authors, such as Brams and Davis. Modifying his notation to fit with ours, his functional 
form measuring the probability that Candidate A wins state \( k \) is:

\[ P_k^a(r_k^a, r_k^b) = \frac{h(r_k^a)}{h(r_k^a) + h(r_k^b)} \] where \( h \) is any twice differentiable function with a positive 
first derivative and negative second derivative for any spending amount greater than or 
equal to zero, and where \( h(0) = 0 \).^{14,15} With this function, the marginal return to 
spending in a state for Candidate A, \( \frac{\partial P_k^a}{\partial r_k^a} \), is greater the less he has already spent. As 
with functions of the logistic nature, the marginal return to spending for Candidate A is 
positive for all possible allocations: \( \frac{\partial P_k^a}{\partial r_k^a} > 0 \quad \forall r_k^a \). The crucial break between this 
functional form and the one we use in this paper is the second derivative. With Snyder’s 
function the return to spending decreases unambiguously as spending increases,

\[ \frac{\partial^2 P_k^a}{\partial (r_k^a)^2} < 0 \quad \forall r_k^a > 0, \] whereas the sign of second derivative for the logistic functional 
form depends on whether \( r_k^a > r_k^b \) or \( r_k^a < r_k^b \). Finally, note that Snyder’s functional form 
is also symmetric in that all of these properties hold when one considers Candidate B’s 
probability of winning the state holding \( r_k^a \) constant.

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14 Snyder, p. 639.

15 Snyder’s actual form includes a term for bias: \( P_k^a(r_k^a, r_k^b) = \frac{c_k^A h(r_k^a)}{c_k^A h(r_k^a) + (1 - c_k^A)h(r_k^b)} \) where 
\( c_k^a \in (0, 1) \) is a measure of the bias in favor of Candidate A in state \( k \). We eliminate this \( (c_k^a = 0.5 \quad \forall k) \), 
which cancels out) for the purposes of comparison since our functional form does not include bias.
Snyder’s function has strange properties when candidates do not spend in a state. If neither candidate allocates any resources to a state the function is undefined. Furthermore, if Candidate Y spends nothing, Candidate X wins the state if they spend any marginally small amount there. If Candidate X spends nothing, Y wins no matter how much he spends. As a means of comparison with our model, we will also analyze parts of the problem using a slight modification of Snyder’s functional form, which retains all of the properties of his function discussed above, but is defined when neither candidate spends in a state:

\[ P_k^a(r_k^a, r_k^b) = \frac{h(r_k^a + 1)}{h(r_k^a + 1) + h(r_k^b + 1)} . \]

4.2 Objective Function

As we noted in the introduction and literature review, there are two main objective functions for this problem. The first is that candidates attempt to maximize their probability of winning the election. The alternative is that candidates try to maximize their expected number of electoral votes. Many authors employ the latter for simplicity and make the claim that the two are equivalent under certain conditions. However, it is easy to see when they might diverge.

Consider a three state election where each state has one electoral vote. If there is no bias and candidates spend equally across states, each candidate has a 50% chance of winning each state and a 50% chance of winning the election. Each candidate’s expected number of electoral votes is 1.5. However, if a candidate could take resources out of one state and distribute them between the remaining two, thereby increasing the probability of winning in those states to 0.72 (with the probability falling to 0 in the other state), his expected number of electoral votes would fall to 1.44, but his probability of winning would increase to 51.8%. Therefore, this would be a useful strategy if the candidate wanted to maximize his chances of winning, but a harmful tactic if he wanted to maximize his expected electoral votes. Lake (1979) has another example of the non-equivalence of these objectives in a case where there are only two states, each with a different number of electoral votes. To maximize their chances of winning, candidates should devote all of their resources to the state with the higher number of votes because
the candidate who wins that state wins the election. However, candidates might want to spend in both states to maximize their expected electoral votes.

Aranson, Hinich, and Ordeshook (1974) present a sufficient condition under which these objectives are equivalent. First, they assume that candidates behave as if they know the expected value of the number of votes they will receive based on their spending decision, \( v_i(\theta) \). The actual number of votes they receive, \( V_i(\theta) \), is a random variable such that \( V_i(\theta) = v_i(\theta) + \varepsilon_i \). They argue that if the expected value of this error term, \( \varepsilon_i \), is zero and the errors are independent of strategies then the objectives are equivalent.

As we showed above this is certainly not always the case. Snyder notes that the objectives only correspond under strict assumptions in the congressional election problem. He writes, “when the game is not so symmetric, and in particular, when one party has an advantage over the other, the two goals may yield rather different equilibria.”

In the end, it does not matter whether a candidate wins the Electoral College by one vote or two hundred votes. The objective is simply to win. Therefore, we assume that both candidates attempt to maximize their probability of winning the election.

Formally, we define a set of states, \( \omega \), as any combination of states, where the cardinality of \( \omega \in \{1, 2, \ldots, n\} \). In order to win the election candidates must win a \textit{winning set of states}, \( \Omega \),

\[
\Omega = \left\{ \omega : \sum_{k \in \omega} \varepsilon_k > \frac{E}{2} \right\}.
\]

The probability that Candidate A wins the election is the sum of the probabilities of Candidate A winning exactly each possible winning set of states:

\[
P_{\text{A wins}} (r_1^a, r_2^a, \ldots, r_n^a, r_1^b, r_2^b, \ldots, r_n^b) = \sum_{\omega \in \Omega} \left( \prod_{k \in \omega} P_k^a \prod_{k \in \omega} (1 - P_k^a) \right)
\]

Since \( P_{\text{A wins}} \) is a sum of the products of continuous functions, \( P_{\text{A wins}} \) is continuous.

With this definition we can formally describe a candidate’s objective in a two-party, \( n \) state election. Given an allocation by his opponent, \( \{r_1^j, r_2^j, \ldots, r_n^j\} \in S_j \),

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\[17\] Snyder, J.M., p. 638.
Candidate I strives to choose a strategy, \( \{r_1^i, r_2^i, \ldots, r_n^i\} \in S_I \), such that he could not obtain a higher probability of winning using any other strategy:

\[
P_{\text{wins}}(r_1^i, r_2^i, \ldots, r_n^i, r_1^j, r_2^j, \ldots, r_n^j) \geq P_{\text{wins}}(r_1^i, r_2^i, \ldots, r_n^i, r_1^j, r_2^j, \ldots, r_n^j) \quad \forall \{r_1^i, r_2^i, \ldots, r_n^i\} \in S_I.
\]

5. Results

5.1 Equilibrium

We first consider the case where candidates have equal total resources, \( R_A = R_B \), and where each state has one electoral vote: \( E = n \). This game is similar to the congressional election game analyzed in Snyder, in which he claims a pure-strategy equilibrium exists under certain conditions. We show that the nature of the logistic functional form measuring the probability that a candidate wins a state eliminates the possibility of a pure-strategy equilibrium at least in this simplified set-up of the game.

**Proposition 1:** No pure-strategy equilibrium exists in this game when \( R_A = R_B \) and \( E = n = 3 \)

**Proof:** If candidates match each other, \( \{r_1^a, r_2^a, \ldots, r_n^a\} = \{r_1^b, r_2^b, \ldots, r_n^b\} \), then

\[
P^a_k(r_k^a, r_k^b) = P^b_k(r_k^a, r_k^b) = 0.5 \quad \forall k \quad \text{and}
\]

\[
P_{\text{wins}}(r_1^a, r_2^a, \ldots, r_n^a, r_1^b, r_2^b, \ldots, r_n^b) = P_{\text{wins}}(r_1^a, r_2^a, \ldots, r_n^a, r_1^b, r_2^b, \ldots, r_n^b) = 0.5.
\]

However, given any strategy by his opponent, a candidate can do better than matching. Assume Candidate A starts by matching his opponent and then withdraws some amount, \( \Delta > 0 \), from a state \( k' \) where \( r_k^b > 0 \) (note that \( \Delta \) must be less than \( r_k^b \)) and divides this amount evenly across the remaining states. WLOG, assume that state \( k' \) is state 1. Therefore, Candidate A exceeds his opponent’s expenditures in states 2 and 3 by \( \Delta/2 \) and trails his opponent’s allocation by \( \Delta \) in state 1. Since Candidate A exceeds Candidate B in states 2 and 3, \( P^a_2(r_2^b + \Delta/2, r_2^b) = P^a_3(r_3^b + \Delta/2, r_3^b) = 0.5 + \delta \) where \( \delta > 0 \).
Furthermore, since the marginal return to spending, $\frac{\partial P^a_k}{\partial r^a_k}$, is greater the closer Candidate A comes to matching Candidate B and the returns are symmetric about $r^b_k$,

$$\frac{\partial P^a_k}{\partial r^a_k} \bigg|_{r^a_k=r^b_k+\Delta} = \frac{\partial P^a_k}{\partial r^a_k} \bigg|_{r^a_k=r^b_k-\Delta},$$

we know that $P^a_1(r^b_1 - \Delta, r^b_1) = 0.5 - \alpha 2 \delta$ where $0 < \alpha < 1$.

Although Candidate A removes twice as much money from state 1 as he reallocates to states 2 and 3, the probability of winning state 1 does not fall by twice as much as the probability of winning rises in states 2 and 3. Therefore,

$$P_{Awins}(r^b_1 - \Delta, r^b_2 + \frac{\Delta}{2}, r^b_3 + \frac{\Delta}{2}, r^b_1, r^b_2, r^b_3)$$

$$= P^a_1 P^a_2 P^a_3 + P^a_1 (1 - P^a_3) + P^a_1 (1 - P^a_2) P^a_3 + (1 - P^a_1) P^a_2 P^a_3$$

$$= (0.5 - 2\alpha \delta)(0.5 + \delta)(0.5 + \delta) + (0.5 + 2\alpha \delta)(0.5 + \delta)(0.5 + \delta)$$

$$+ (0.5 - 2\alpha \delta)(0.5 + \delta)(0.5 - \delta) + (0.5 - 2\alpha \delta)(0.5 - \delta)(0.5 + \delta)$$

$$= (0.5 + \delta)[(0.5 - 2\alpha \delta)(0.5 + \delta) + (0.5 + 2\alpha \delta)(0.5 + \delta) + 2(0.5 - 2\alpha \delta)(0.5 - \delta)]$$

$$= (0.5 + \delta)[0.25 + 0.5\delta - \alpha \delta - 2\alpha \delta^2 + 0.25 + 0.5\delta + \alpha \delta + 2\alpha \delta^2 + 0.5 - \delta - 2\alpha \delta + 4\alpha \delta^2]$$

$$= (0.5 + \delta)[1 - 2\alpha \delta + 4\alpha \delta^3]$$

$$= 0.5 + (1 - \alpha)\delta + 4\alpha \delta^3$$

It is easy to see that $P_{Awins}(r^b_1 - \Delta, r^b_2 + \frac{\Delta}{2}, r^b_3 + \frac{\Delta}{2}, r^b_1, r^b_2, r^b_3) > 0.5$ for $\delta > 0$ and $0 < \alpha < 1$. Therefore, Candidate A prefers this allocation to a matching strategy. However, in this case $P_{Bwins}(r^a_1, r^a_2, r^a_3, r^b_1, r^b_2, r^b_3) = 1 - P_{Awins}(r^a_1, r^a_2, r^a_3, r^b_1, r^b_2, r^b_3) < 0.5$. This cannot be an equilibrium since Candidate B could employ the same strategy Candidate A used, given Candidate A’s new allocation, such that

$$P_{Bwins}(r^a_1, r^a_2, r^a_3, r^b_1, r^b_2, r^b_3) > 0.5$$. This would reduce $P_{Awins}(r^a_1, r^a_2, r^a_3, r^b_1, r^b_2, r^b_3) < 0.5$.

Since a candidate can always choose an allocation based on his opponent’s allocation to increase his probability above 0.5 and drive his opponent’s to less than 0.5, no pure-strategy equilibrium exists.
5.2 Snyder’s Equilibrium

We now consider the modification of Snyder’s functional form,

\[ P_k^a (r_k^a, r_k^b) = \frac{h(r_k^a + 1)}{h(r_k^a + 1) + h(r_k^b + 1)} , \]

as a means of comparison. Snyder claims in Proposition 4.2 that if \( x^b \leq b \leq \frac{1}{n} \) then a unique pure-strategy equilibrium exists where both players spend some positive amount in every state. This model has many similarities with the problem addressed in this paper and it is worth examining why this pure-strategy equilibrium emerges with Snyder’s function, but fails to surface using the logistic functional form to measure the probability that a candidate wins a particular state. The crucial difference between the two models is the concavity of the candidates’ objective functions. In this section we explain the importance of this concavity for an equilibrium using the Debreu-Glicksberg-Fan Theorem (Debreu (1952); Glicksberg (1952); Fan (1952)), and discuss the specific characteristics of the different functional forms that yield concavity in Snyder’s case, but not in ours.

The model with the logistic functional form and the model with Snyder’s functional form share many of the same properties. We know the objective function in each model is continuous for both candidates in each of their strategies given any allocation by their opponent because it is the sum of the products of functions which are continuous. Furthermore, it is easy to see that each candidate’s strategy set is an n-dimensional simplex where the sum of the allocation vector’s components is \( R_i \) rather than one. This set is nonempty (as long as \( R_i > 0 \)), convex and compact. Therefore, we know a pure strategy equilibrium exists by the Debreu-Glicksberg-Fan Theorem, if the objective functions for both players are concave in their strategies given any resource allocation by their opponent. Snyder demonstrated that his model meets this condition if \( h(x) = x^b \) and \( b \leq \frac{1}{n} \). However, the objective functions in the logistic model are not

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18 Snyder, p. 650. This claim is based on his original functional form, but the proof still holds with the modification.
concave. The following graphs emphasize the differences between the objective functions derived from the different functional forms for the probability of winning a state:

*Figure 2*: Candidate A’s objective function, \( P_{\text{wins}}(r_1^a, r_1^b, r_2^a, 2, 3, 5) \), where \( R_A = R_B = 10 \) in the model with the Logistic functional form

*Figure 3*: Candidate A’s objective function, \( P_{\text{wins}}(r_1^a, r_1^b, r_2^a, 2, 3, 5) \), where \( R_A = R_B = 10 \) in the model with Snyder’s functional form

The differences in concavity arise from the differences in the second derivatives of the probability of winning functions. As we discussed in section 4.1,

\[
\frac{\partial^2 P_k^a}{\partial (r_k^a)^2} > 0 \quad \forall r_k^a < r_k^b \quad \text{and} \quad \frac{\partial^2 P_k^a}{\partial (r_k^a)^2} < 0 \quad \forall r_k^a > r_k^b \quad \text{when} \quad P_k^a \quad \text{is measured by the logistic function.} \]

A candidate’s initial marginal return to spending in a state can be quite small. As we demonstrated in the proof of Proposition 1, this allows a candidate to improve his chances of winning by removing some resources from one state and dividing them between the remaining states. Reciprocally, a candidate does worse by removing an equal amount of resources from \( n - 1 \) states and distributing them to the remaining state. These properties create saddle-like non-concave objective functions like the one in Figure 2.

However, with Snyder’s function, \( \frac{\partial^2 P_k^a}{\partial (r_k^a)^2} < 0 \quad \forall r_k^a > 0 \). A candidate receives the highest marginal return on the first dollars he spends in a state, and this return falls the more he spends. Therefore, it is not necessarily the case that a candidate can do better by spending \( \Delta \) less than his opponent in one state and distributing it amongst the remaining states. Consider the case where both candidates are allocating their resources equally across states. In the three state case, if Candidate A removes \( \Delta \) amount of resources from
state 1 and distributes it evenly across the other two states, then

\[ P_2^a(r_2^b + \frac{\Delta}{2}, r_2^b) = P_3^a(r_3^b + \frac{\Delta}{2}, r_3^b) = 0.5 + \delta \] where \( \delta > 0 \) and

\[ P_1^a(r_1^b - \Delta, r_1^b) = 0.5 - \alpha 2\delta \] where \( \alpha > 1 \). \( \alpha \) is greater than one in this case because the return to spending in state 1 is higher than in states 2 and 3:

\[
\frac{\partial P_1^a}{\partial r_1^a} \bigg|_{r_1^a = r_1^b - \Delta} > \frac{\partial P_2^a}{\partial r_2^a} \bigg|_{r_2^a = r_2^b - \Delta/2} = \frac{\partial P_3^a}{\partial r_3^a} \bigg|_{r_3^a = r_3^b - \Delta/2}
\]

The opposite is the case with the logistic model.

The end result is:

\[ P_{\text{wins}}(r_1^b - \Delta, r_2^b + \frac{\Delta}{2}, r_3^b + \frac{\Delta}{2}, r_1^b, r_2^b, r_3^b) \]

\[ = 0.5 + (1 - \alpha)\delta + 4\alpha\delta^3 \]

However, since \( \alpha > 1 \), \( P_{\text{wins}}(r_1^b - \Delta, r_2^b + \frac{\Delta}{2}, r_3^b + \frac{\Delta}{2}, r_1^b, r_2^b, r_3^b) \) is only greater than 0.5 if \( \alpha < \frac{1}{1 - 4\delta^2} \), which need not be the case. Therefore, the proof of non-existence used for the logistic functional form fails with Snyder’s form. We know that it can not work since a pure-strategy equilibrium exists by the Debreu-Glicksberg-Fan Theorem; however, it is useful to see how the nature of the different functional forms affect \( \alpha \).

5.3 Mixed Strategy Equilibrium

We demonstrated in section 5.1 that no pure-strategy equilibrium exists in a simplified version of our model. A natural question which follows is whether a mixed strategy equilibrium exists. Glicksberg (1952) demonstrates that a mixed strategy equilibrium exists in games that are compact, take place in Hausdorff space, and have continuous payoff functions. This game is compact because the candidate’s strategy sets are compact and nonempty (as long as \( R_j > 0 \)), and the payoff functions are bounded (the probability of winning lies between 0 and 1). Furthermore, the topological space is the real numbers, which is a Hausdorff space. Finally, we noted in section 4.2 that each

candidate’s payoff function is continuous given any strategy by their opponent because it is the sum of the products of continuous functions. Therefore, this game has a mixed strategy equilibrium.

5.4 Best Response Function

Another question of interest is how a candidate can optimally allocate his resources given his opponent’s strategy. Generally, we define Candidate A’s Best Response Function as:

\[ BR_A (r_1^b, r_2^b, ..., r_n^b) = \{ \{ r_1^a, r_2^a, ..., r_n^a \} \in S_A : P_{\text{wins}} (r_1^a, r_2^a, ..., r_n^a, r_1^b, r_2^b, ..., r_n^b) \geq P_{\text{wins}} (r_1^*, r_2^*, ..., r_n^*) \} \]

In words, Candidate A’s best response to any recourse allocation, \( \{ r_1^b, r_2^b, ..., r_n^b \} \in S_B \), by his opponent, is to choose the spending vector \( \{ r_1^a, r_2^a, ..., r_n^a \} \in S_A \) which maximizes his objective function \( P_{\text{wins}} \). As in any constrained maximization problem, Candidate A’s possible best responses are the interior critical points of \( P_{\text{wins}} \) where

\[
\frac{\partial P_{\text{wins}}}{\partial r_1^a} = \frac{\partial P_{\text{wins}}}{\partial r_2^a} = ... = \frac{\partial P_{\text{wins}}}{\partial r_n^a} \quad \text{and the critical points along the constraint}
\]

where \( \frac{\partial P_{\text{wins}}}{\partial r_k^a} = \frac{\partial P_{\text{wins}}}{\partial r_{k'}^a} \forall r_k^a, r_{k'}^a > 0 \). If Candidate A is not spending at a critical point then

\[
\frac{\partial P_{\text{wins}}}{\partial r_k^a} > \frac{\partial P_{\text{wins}}}{\partial r_{k'}^a} \quad \text{for at least one } k \text{ and } k' \text{ where } r_k^a, r_{k'}^a > 0 , \quad \text{and the candidate could do better by shifting resources from state } k' \text{ to state } k.
\]

The following contour plots of Candidate A’s objective function show examples of the candidate’s best response when \( n = 3 \) and \( R_A = R_B = 10 \).
Figure 4: Contour plot of Candidate A’s objective function for n=3 when Candidate B spends on all three states ( \{ r_1^b, r_2^b, r_3^b \} = \{3, 3, 4\} ).

Figure 5: Contour plot of Candidate A’s Objective function for n=3 when Candidate B spends on two of the three states ( \{ r_1^b, r_2^b, r_3^b \} = \{0, 3, 7\} ).
We hypothesize that if candidates have equal resources, $R_A = R_B$,

$$BR_A(r^a_1, r^a_2, ..., r^a_n) \in Bnd(S_A) \quad \forall \{r^b_1, r^b_2, ..., r^b_n\} \in S_B.$$ 

Candidate A’s best response is always to spend zero in at least one state. By not spending in this state(s), Candidate A has additional resources to exceed his opponent in the remaining states. As we demonstrated in Proposition 1, removing $\Delta$ amount of resources from one state after matching the opponent and distributing this excess evenly across the other states increases a candidate’s probability of winning a three state election. At least in the three state case, Candidate A’s best response is to consider matching his opponent and proceed to remove $\Delta = r^b_k$ from a state $k'$ where $r^b_k \geq r^b_k \quad \forall \, k$ and distribute this $\Delta$ amount of resources evenly across the other two states. This implies that Candidate A spends the most in the state where his opponent allocates the second most. In models with higher numbers of states it may be optimal for Candidate A to remove resources from more than one state (but still states where Candidate B is spending the most) and distribute them amongst the
remaining states. An interesting topic to explore is when it is optimal for a Candidate to ignore more than one state.

Discontinuities in the best response function arise from the hypothesis that Candidate A’s best response is to avoid spending in the state where Candidate B is spending the most, and instead allocate the most resources to the state where his opponent is spending the second most. Small changes in Candidate B’s allocation that alter the state where he is spending the most lead Candidate A to abandon one state in favor of another. For example, in a three state model, if Candidate B initially allocates the most resources to state 1 and the rest to state 2, Candidate A wants to spend nothing in the first state and the majority of his resources in second state. However, if Candidate B shifts his allocation slightly between states 1 and 2, such that state 2 becomes the state where he allocates the most, Candidate A suddenly wants to spend nothing in state two and instead put the majority of his resources into state 1. The following diagram shows Candidate A’s best responses given a finite number of possible allocations, \( \{ r_1^b, r_2^b, ..., r_n^b \} \), by Candidate B (there are three states total and both candidates have 10 total resources):

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\( \text{Figure 7: Best Responses for Candidate A given allocations by Candidate B.} \)

Yellow highlights the region where Candidate B spends the most in state 3; the red region is where Candidate B spends the most in state 2; the light blue region indicates that Candidate B spends the most on state 1. The other colors mean that Candidate B spends the most on multiple states (Candidate A has multiple best responses at these points). The thick black lines indicate the discontinuities in Candidate A’s best response.
function. As we explained above, these breaks occur between regions where the state in which Candidate B allocates the most resources shifts.

5.5 Unequal Resources

Another area that we have begun to explore is how the nature of a candidate’s best response function changes with unequal resources. In contrast with the equal resources case, if the inequality is large enough it is optimal for the wealthier candidate to spend on all three states and for the poorer candidate to devote all of his resources to one state in certain situations. Consider the case where Candidate A has 25 total resources and Candidate B has 10. The following figure shows the states where Candidate A spends to maximize his objective function given different allocations by Candidate B:

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*Figure 8:* The states where Candidate A spends positive resources to maximize his objective function given different strategies by Candidate B

Candidate A’s best response is to spend on all three states when Candidate B’s expenditures are spread fairly evenly across states (middle of the diagram). This is
preferable to spending on two states because of the nature of the marginal returns within a state. Since Candidate A has significantly more resources, his marginal returns to spending in just two states become quite small as he exceeds his opponent by large amounts. He is better off accepting the initial lower returns in the third state in order to also catch up and exceed his opponent there. The following figure shows the reciprocal diagram for the poorer Candidate B, given various allocations by his wealthier opponent:

| $r_2^*$ | 0 | 1/18 | 1/9 | 1/6 | 2/9 | 5/18 | 1/3 | 7/18 | 4/9 | 1/2 | 5/9 | 11/18 | 2/3 | 13/18 | 7/9 | 5/6 | 8/9 | 17/18 | 1 |
|--------|---|------|-----|-----|-----|------|-----|------|-----|-----|-----|--------|-----|--------|-----|-----|-----|------|  |
| 0      | 162 | 162 | 162 | 162 | 162 | 162 | 162 | 2    | 2   | 2   | 2   | 153   | 153 | 153   | 153 | 153 | 153 | 153   |  |
| 1/18   | 162 | 162 | 162 | 162 | 162 | 162 | 162 | 162  | 162 | 162 | 162 | 162   | 162 | 162   | 162 | 162 | 162 | 162   |  |
| 1/9    | 162 | 162 | 162 | 162 | 162 | 162 | 162 | 162  | 162 | 162 | 162 | 162   | 162 | 162   | 162 | 162 | 162 | 162   |  |
| 1/6    | 162 | 162 | 162 | 162 | 162 | 162 | 162 | 162  | 162 | 162 | 162 | 162   | 162 | 162   | 162 | 162 | 162 | 162   |  |
| 2/9    | 162 | 162 | 162 | 162 | 162 | 162 | 162 | 162  | 162 | 162 | 162 | 162   | 162 | 162   | 162 | 162 | 162 | 162   |  |
| 5/18   | 162 | 162 | 162 | 162 | 162 | 162 | 162 | 162  | 162 | 162 | 162 | 162   | 162 | 162   | 162 | 162 | 162 | 162   |  |
| 1/3    | 162 | 162 | 162 | 162 | 162 | 162 | 162 | 162  | 162 | 162 | 162 | 162   | 162 | 162   | 162 | 162 | 162 | 162   |  |
| 7/18   | 162 | 162 | 162 | 162 | 162 | 162 | 162 | 162  | 162 | 162 | 162 | 162   | 162 | 162   | 162 | 162 | 162 | 162   |  |
| 4/9    | 1  | 162 | 162 | 263 | 263 | 263 | 263 | 263  | 263 | 263 | 263 | 263   | 153 | 153   | 153 | 153 | 153 | 153   |  |
| 1/2    | 1  | 263 | 263 | 263 | 263 | 263 | 263 | 263  | 263 | 263 | 263 | 263   | 1  | 153   | 153 | 153 | 153 | 153   |  |
| 5/9    | 3  | 263 | 263 | 263 | 263 | 263 | 263 | 263  | 263 | 263 | 263 | 263   | 2  | 1     | 153 | 153 | 153 | 153   |  |
| 11/18  | 263 | 263 | 263 | 263 | 263 | 263 | 263 | 263  | 263 | 263 | 263 | 263   | 1  | 1     | 1    | 153 | 153 | 153   |  |
| 2/3    | 263 | 263 | 263 | 263 | 263 | 263 | 263 | 263  | 263 | 263 | 263 | 263   | 263 | 263   | 1    | 153 | 153 | 153   |  |
| 13/18  | 263 | 263 | 263 | 263 | 263 | 263 | 263 | 263  | 263 | 263 | 263 | 263   | 263 | 263   | 263  | 153 | 153 | 153   |  |
| 7/9    | 263 | 263 | 263 | 263 | 263 | 263 | 263 | 263  | 263 | 263 | 263 | 263   | 263 | 263   | 263  | 263 | 153 | 153   |  |
| 5/6    | 263 | 263 | 263 | 263 | 263 | 263 | 263 | 263  | 263 | 263 | 263 | 263   | 263 | 263   | 263  | 263 | 263 | 153   |  |
| 8/9    | 263 | 263 | 263 | 263 | 263 | 263 | 263 | 263  | 263 | 263 | 263 | 263   | 263 | 263   | 263  | 263 | 263 | 153   |  |
| 17/18  | 263 | 263 | 263 | 263 | 263 | 263 | 263 | 263  | 263 | 263 | 263 | 263   | 263 | 263   | 263  | 263 | 263 | 153   |  |
| 1      | 263 | 263 | 263 | 263 | 263 | 263 | 263 | 263  | 263 | 263 | 263 | 263   | 263 | 263   | 263  | 263 | 263 | 153   |  |

*Figure 9:* The states where Candidate B spends positive resources to maximize his objective function given different strategies by Candidate A

If Candidate A splits his superior resources fairly evenly between two states, Candidate B’s best response is to throw all of his resources into one of those two states.

**5.6 Bias**

There are multiple possibilities for including bias in the logistic functional form, each of which may have a different impact on the results. First, one could modify the
functional form such that \( P_k^a(r_k^a, r_k^b, c_k) = \frac{1}{1 + e^{-(r_k^b - r_k^a + c_k)}} \), where \( c_k \) is a measure of the bias. This method preserves all the characteristics of the functional form but shifts the inflection point (where \( P_k^a(r_k^a, r_k^b) = 0.5 \) and \( \frac{\partial^2 P_k^a}{\partial (r_k^a)^2} = 0 \) ) left (if \( c_k > 0 \) ) or right (if \( c_k < 0 \)). Candidate A achieves equal chances of winning the state by spending \( c_k \) less (or more if \( c_k < 0 \)) than his opponent. The larger \( c_k \), the greater the bias in favor Candidate A. If \( c_k \) is negative Candidate B has an advantage.

A second idea is to preserve the inflection point at \( r_k^a = r_k^b \), but shift and compress the probability of winning function vertically. For example, let

\[
P_k^a(r_k^a, r_k^b, c_k) = \frac{1 - |c_k|}{1 + e^{-(r_k^b - r_k^a)}} + \frac{|c_k| + c_k}{2}, \text{ where } -1 < c_k < 1.
\]

The closer \( c_k \) is to one, the greater the advantage for Candidate A; the closer \( c_k \) is to negative one the larger the edge for Candidate B. Unlike the first possibility, this functional form reduces the marginal returns for any positive value of spending as the bias in favor of one candidate increases:

\[
\frac{\partial P_k^a}{\partial r_k^a} < 0. \text{ It also restricts the range of the probability to less than } (0, 1).
\]

6. Conclusion

In this paper we have attempted to provide further insight into the problem of campaign resource allocation through the use of a different functional form for the probability that a candidate wins a particular state. We have shown that there is no pure-strategy equilibrium in simple cases of the model, which breaks with much of the previous literature on the subject. In comparison to Snyder, this arises because of the differences in the second derivatives of the functions describing candidates’ probabilities of winning a state. These derivatives help yield concavity in Snyder’s objective functions, but lead to saddle-like objective functions in our model. Clearly there is still a great deal to be explored, such as generalizing the results to \( n \) states with different numbers of
electoral votes, understanding how the best response functions interact in the unequal resources case, and examining the issue of bias.
7. Reference List


Federal Election Commission. Retrieved May 1, 2008, from:  


